1,7 Metric Equation

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Metric Equation in Absence of Gravity

The material in this article was taken from *Believe* [1], Chapter 8. Several relativistic effects produced by gravity are derived quite simply from the metric equation, which is based on the elements of the metric tensor. The following discussion explains the principles of the metric equation.

Consider two points, denoted (1), (2), separated by distances of Δx, Δy, Δz in the x, y, z directions. The Pythagorean Theorem shows that the distance Δd between the two points is calculated from:

$$(\Delta d)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \qquad (1)$$

Assume that a light pulse is generated at point (1) and propagates to point (2) in a time denoted Δt. The distance Δd between the two points is equal to the speed of light c multiplied by the time interval Δt. Since Δd is equal to $c\Delta t$, the following equation holds for this experiment:

$$0 = (c\Delta t)^2 - (\Delta d)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \qquad (2)$$

To handle more general experiments, the difference between $(c\Delta t)^2$ and $(\Delta d)^2$ is defined as $(\Delta s)^2$, and the equation becomes

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta d)^2 = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \qquad (3)$$

This is called the ***metric equation***.

A relativistic occurrence is defined in terms of an ***event***, which is characterized by the ***time*** of the event and the ***three-dimensional spatial coordinates*** that indicate the point where the event occurs. The generation of the light pulse at point (1) is defined as event (1), and the reception of the light pulse at point (2) is defined as event (2). In this experiment, the quantity $(\Delta s)^2$ for the metric equation is zero.

Now let us assume that a missile is fired from point (1) to point (2). Event (1) represents the firing of the missile at point (1), and event (2) represents the impact of that missile at point (2). Since the speed of the missile is less than the speed of light, the time difference Δt between the two events is greater than in the first experiment. Consequently $(\Delta s)^2$ is greater than zero. ***A situation where $(\Delta s)^2$ is greater than zero is called time-like; in a time-like situation, one event can influence the other.*** The first experiment, where $(\Delta s)^2$ is zero, is the limiting case of the ***time-like*** condition.

Let us assume that point (2) has an early-warning system, which detects the flash of the missile firing at point (1). In response to this flash, point (2) fires a missile in return. Event (1) is defined as the firing of the first missile from point (1) and event (2) is defined as the firing of the second missile from point (2). The difference between these two events is still ***time-like***. The quantity $(\Delta s)^2$ is greater than zero, and so one event can (and did) influence the other.

Now assume that two warring parties have a truce that ends at 12:00 noon. Not trusting the other party, each one decides to launch a preemptive strike exactly at noon. Event (1) is when point (1) fires its missile and event (2) is when point (2) fires its missile. In this case $(\Delta s)^2$ is negative, and the condition is called ***space-like***. ***If*** $(\Delta s)^2$ ***is negative, so that the interval between two events is space-like, there is not enough time between the two events for a light pulse to travel between them. Consequently one event cannot influence the other in a space-like situation.***

Let us apply the metric equation to the space travel experiment discussed in Chapter 6 of *Believe* [1], where two observers travelling at different speeds are measuring the speed of light. The two observers will measure different values for the time and spatial variables involved in the experiment. However if $(\Delta s)^2$ is zero for one observer, it is zero for the other, because the speed of light is the same for both observers. It can also be shown that the two observers always measure the same value for $(\Delta s)^2$, and so the quantity $(\Delta s)^2$ is called ***invariant***. By recognizing that $(\Delta s)^2$ is the same for all observers, one can derive the equations for the Lorentz transformation, which characterizes the relations of special relativity discussed in Chapter 6 of *Believe* [1].

Effect of Gravity on Metric Equation

In the presence of a gravitational field, the speed of light is not constant, and so the simple metric equation no longer applies and must be generalized. To achieve this, the simple metric equation is first expressed in terms of the general coordinates x_0, x_1, x_2, x_3. Coordinate x_0 represents normalized time, which is equal to ct and is denoted τ. Coordinates x_1, x_2, x_3, represent the spatial coordinates, which are x, y, z in rectangular coordinates. The metric equation becomes

$$(\Delta s)^2 = (\Delta\tau)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (\Delta x_0)^2 - (\Delta x_1)^2 - (\Delta x_2)^2 - (\Delta x_3)^2 \quad (4)$$

In a gravitational field, the terms of the metric equation are modified by the metric tensor elements g_{00}, g_{11}, g_{22}, g_{33} to give:

$$\begin{aligned}(\Delta s)^2 &= g_{00}(\Delta x_0)^2 + g_{11}(\Delta x_1)^2 + g_{22}(\Delta x_2)^2 + g_{33}(\Delta x_3)^2 \\ &= g_{00}(\Delta\tau)^2 + g_{11}(\Delta x)^2 + g_{22}(\Delta y)^2 + g_{33}(\Delta z)^2 \qquad (5)\end{aligned}$$

This assumes that the metric tensor is diagonal and so has only four elements. For a nondiagonal metric tensor, which has 16 elements, the metric equation has 16 terms. The metric equation for a nondiagonal metric tensor has cross-product terms, such as $g_{13}\Delta x_1\Delta x_3$. We will restrict our discussion to diagonal metric tensors. After all, Einstein could consider only diagonal metric tensors; otherwise his equations would have been too complicated to be solved analytically.

Let us compare the general metric formula of equation 5 with that of equation 4, which applies in the absence of a gravitational field. This comparison shows that with no gravitational field the metric tensor element g_{00} is equal to +1 and the elements g_{11}, g_{22}, g_{33} are equal to -1. Within our solar system the gravitational field is weak, and so the values of the metric tensor elements are very close to these ideal values, which apply in special relativity conditions. This is illustrated in Table 1.

Table 1: Values of metric tensor elements in rectangular coordinates for zero gravitational field (special relativity) and the approximate values that apply within our solar system.

element	zero gravity	approximation in solar system
g_{00}	1	$1 - 2(m/r)$
$g_{11} = g_{22} = g_{33}$	-1	$-1 - 2(m/r)$

Table 1 gives the metric tensor elements for no gravitational field, and the approximate values that apply within our solar system. The values of m/r within our solar system are very small. Its maximum value is only $2.12\text{x}10^{-6}$, which occurs at the surface of the sun. The maximum value for 2(m/r) is only $4.24\text{x}10^{-6}$, or 4.24 parts per million. Therefore Table 1 shows that the metric tensor elements within our solar system are very close to the special-relativity values, which apply with no gravitational field.

Calculating Relativistic Effects from Metric Equation

We can easily derive some important principles of general relativity from the metric equation. These include the effect of a gravitational field on the speed of light, the contraction of a distance interval, and the expansion of a time interval. With the Yilmaz theory, the metric tensor elements g_{11}, g_{22}, g_{33} in rectangular coordinates are equal. Consequently the speed of light and the contraction of distance is the same for all directions. This allows us to simplify the metric equation by considering a single spatial dimension, which we call x. Hence the metric equation reduces to

$$(\Delta s)^2 = g_{00}(\Delta\tau)^2 + g_{11}(\Delta x)^2 \qquad (6)$$

For the Schwartzschild solution of the Einstein theory, the speed of light and the contraction of distance are not the same in all directions. This complicated issue indicates a deficiency in the Schwartzschild solution. It is discussed in Appendix G of *Believe* [1], but we now ignore it. For the Schwartzschild solution, we restrict our present attention to motion in the radial direction, and we ignore motion perpendicular to the radius.

If Δs in the metric equation is set equal to zero, a light pulse can just travel between the two events. Hence this condition yields the speed of light.

If Δx is set equal to zero in the metric equation, $(\Delta s)^2$ is positive, because the metric tensor element g_{00} is positive. Therefore the interval between the two events is ***time-like***. In this case the value for Δs is called the ***proper time*** between the two events, which we denote $\Delta\tau_p$. ***Proper time is the time difference read by a clock that is moved at constant velocity between the two events.*** (In this website, “proper time” is usually called “true time”.)

If $\Delta\tau$ is set equal to zero in the metric equation, $(\Delta s)^2$ is negative, because the metric tensor element g_{11} is negative. Therefore the interval between the two events is ***space-like***. In this case the value for $\sqrt{[-(\Delta s)^2]}$ is called the ***proper distance*** between the two events, which we denote Δx_p. ***Proper distance is the distance read on a ruler stretched between the two events, the ruler being at rest in the coordinate system for which the two events are simultaneous.*** (In this website, “proper distance” is usually called “true distance”.)

Applying these principles to the metric equation gives the results shown in Table 2. In the first case, Δs is set to zero, and the ratio $\Delta x/\Delta\tau$ is calculated, which (as we will see) gives the apparent relative speed of light. In the second case, Δx is set to zero, and the value for Δs is the proper time, which is denoted $\Delta\tau_p$. In the third case, $\Delta\tau$ is set to zero, and the value for $\sqrt{[-(\Delta s)^2]}$ is the proper distance, which is denoted Δx_p. The formulas that are calculated from the metric equation for the variables shown in the second column are given in the third column.

Table 2: Derivation of relativistic parameters from simplified metric equation

condition	variable	formula	meaning
$\Delta s = 0$	$\Delta x/\Delta\tau$	$\sqrt{[-g_{00}/g_{11}]}$	speed of light
$\Delta x = 0$	$\Delta\tau_p = \Delta s$	$\sqrt{[g_{00}]}\ \Delta\tau$	proper time
$\Delta\tau = 0$	$\Delta x_p = \sqrt{[-(\Delta s)^2]}$	$\sqrt{[-g_{11}]}\ \Delta x$	proper distance

The variables $\Delta\tau$ and Δx in Table 2 are usually called ***coordinate time*** and ***coordinate distance***. These are time and distance intervals that are measured at infinity relative to a star, and so are the values experienced by an observer on earth. We use the term "apparent" to represent the term that is usually called "coordinate" in the relativity literature. The variables $\Delta\tau$ and Δx in Table 8-2 are called ***apparent time*** and ***apparent distance*** intervals, and are denoted $\Delta\tau_{ap}$ and Δx_{ap}.

Since $\Delta\tau$ is equal to $c\Delta t$, the ratio $\Delta x/\Delta\tau$ is equal to $(1/c)(\Delta x/\Delta t)$. The velocity $\Delta x/\Delta t$ for this case is the apparent speed of light, which we denote c_{ap}. Hence the ratio $\Delta x/\Delta\tau$ is equal to c_{ap}/c, which is the relative value of the apparent speed of light.

The above principles are applied to Table 2 to give the ratios shown in Table 3. The first column shows the variables in terms of physically meaningful terms, and the second column shows the values specified in the metric equation. The third column gives the formulas for these ratios that are calculated from the metric equation. A gravitational field causes the speed of light to decrease, it causes a dimension to contract, and it causes a time interval to expand. Because of the expansion of time interval, a clock runs slower.

Table 3: Relativistic parameters derived from metric equation.

ratio physical	ratio equation	formula	
c_{ap}/c	$\Delta x/\Delta\tau$	$\surd[-g_{00}/g_{11}]$	(speed of light)
$\Delta x_{ap}/\Delta x_p$	$\Delta x/\surd[-(\Delta s)^2]$	$1/\surd[-g_{11}]$	(spatial dimension)
$\Delta\tau_{ap}/\Delta\tau_p$	$\Delta\tau/\Delta s$	$1/\surd[g_{00}]$	(clock period)

The parameter (τ) represents relativistic time (ct), which is the speed of light (c) multiplied by actual time (t). Hence the ratio ($\Delta\tau_{ap}/\Delta\tau_p$) is equal to the ratio (T_{ap}/T_p), where (T_{ap}) is the apparent period of a clock in actual time, and (T_p) is the proper, or "true", period of the clock in actual time. The subscript (p) is usually omitted by the author, and so this clock period ratio is expressed as

$$T_{ap}/T = 1/\surd[g_{00}] \tag{7}$$

The "true" or "proper" clock period (T) is the value measured at the location in the gravitational field for the specified value of (g_{00}), and the apparent clock period (T_{ap}) is measured at a distant location far from this gravitational field.

References

[1] Adrian Bjornson, *A Universe that We Can Believe*, Addison Press, (www.olduniverse.com), 2000, ISBN 09703231-0-7.