5,2 Addendum Chapter 2
Schwartzschild and Isotropic Solutions of the Einstein Theory

Adrian Bjornson (May 2009)

This chapter describes the derivation of two commonly used solutions of the Einstein theory: the Schwartzschild solution and the related isotropic solution. These describe the gravitational effects of a star. The Schwartzschild solution is non-isotropic, and so it predicts that the velocity of light is different in the radial and tangential directions, relative to the center of the star. An even stranger property is that spatial compression is different in the radial and tangential directions. Consequently, the ratio of circumference to diameter of a circle centered about the star is not equal to π; it is greater than π.

The isotropic solution does not have this strange property of the Schwartzschild solution. However, the interior isotropic solution does not yield an analytical answer for pressure inside the star, and so the theoretical foundation for the isotropic solution is weak. It should be possible to remedy this with numerical computer analysis, but this apparently has not been done

1, Description of Schwartzschild Solution

Schwartzschld began his analysis by assuming that the metric equation is diagonal in spherical coordinates and has the following elements:

$$g_{00} = e^{\nu} \; ; \; g_{11} = - e^{\lambda} \; ; \; g_{22} = - r^2 \; ; \; g_{33} = - (r \sin \theta)^2 \qquad (1)$$

The general form of the metric equation in spherical coordinates is

$$ds^2 = g_{00}(d\tau)^2 + g_{11}(dr)^2 + g_{22}(d\theta)^2 + g_{33}(d\psi)^2 \qquad (2)$$

The variables ν and λ are assumed to vary only with radial distance r.

Section 5 of *5,B Addendum Appendix B* gives formulas for computing the elements of the Einstein tensor $G_\mu{}^\nu$ from the metric tensor, when the metric tensor is diagonal. These were originally derived by Dingle and were published by Tolman [5]. With these formulas one can calculate from the metric tensor values in Eq. 1 the following formulas for the non-zero elements of the corresponding Einstein tensor:

$$G_0{}^0 = e^{-\lambda} \{ - (\lambda'/r) + (1/r^2) \} - (1/r^2) \qquad (3)$$

$$G_1{}^1 = e^{-\lambda} \{ (\nu'/r) + (1/r^2) \} - (1/r^2) \qquad (4)$$

$$G_2{}^2 = G_3{}^3 = e^{-\lambda} \{ ½ (\nu'/r) - ½ (\lambda'/r) + ½ \nu'' - ¼ \lambda' \nu' + ¼ (\nu')^2 \} \qquad (5)$$

These formulas are given by Tolman [5] (p. 242, Eq. 95.3). The variables ν', λ' denote the partial derivatives of ν, λ relative to radial distance r. The variable ν'' is the second partial derivative of ν relative to radial distance r, which is obtained by performing two partial-derivative computations in succession. The Einstein tensor symbol $G_\mu{}^\nu$ was not used by Tolman; instead Tolman employed the expression $-8\pi T_\mu{}^\nu$ to represent $G_\mu{}^\nu$ in his equations.

The Schwartzschild analysis has two solutions: (1) the Schwartzschild interior solution, which applies inside the star, and (2) the Schwartzschild exterior solution, which applies in the vacuum of space outside the star. Let us first consider the interior solution.

Schwartzschild interior solution. The analysis performed by Schwartzschild to calculate the energy-momentum tensor for the interior of a star is presented in *5,C Addendum Appendix C*. He assumed that the star is a perfect fluid, with constant mass density and no viscous (or shear) forces. As shown in Eq. 30 of 5,C the resultant energy-momentum tensor is diagonal and has the following nonzero elements:

$$T_0{}^0 = \rho \quad ; \quad T_1{}^1 = T_2{}^2 = T_3{}^3 = -p \tag{6}$$

The parameter ρ is the mass density of the star, which Schwartzschild assumed to be constant, and p is pressure within the star, which varies with radius. The density and pressure are expressed in normalized relativistic units. The Einstein gravitational field equation is

$$G_\mu{}^\nu = -8\pi\, T_\mu{}^\nu \tag{7}$$

Applying this gravitational field equation to the values in Eq. 6, gives the following for the required elements of the Einstein tensor $G_\mu{}^\nu$:

$$G_0{}^0 = -8\pi\rho \quad ; \quad G_1{}^1 = G_2{}^2 = G_3{}^3 = 8\pi p \tag{8}$$

Substituting these values into Eqs. 3 to 5 gives the following equations for the Schwartzschild interior solution, which holds for ($r <= r_s$), where r_s is the radius of the star:

$$G_0{}^0 = -8\pi\rho = e^{-\lambda}\,\{ -(\lambda'/r) + (1/r^2) \} - (1/r^2) \tag{9}$$

$$G_1{}^1 = 8\pi p = e^{-\lambda}\,\{ (\nu'/r) + (1/r^2) \} - (1/r^2) \tag{10}$$

$$G_2{}^2 = G_3{}^3 = 8\pi p = e^{-\lambda}\{½\,(\nu'/r) - ½\,(\lambda'/r) + ½\,\nu'' - ¼\,\lambda'\,\nu' + ¼\,(\nu')^2\} \tag{11}$$

Schwartzschild exterior solution. In the vacuum of space outside the star, the energy momentum tensor is identically zero, and so the Einstein gravitational field equation requires that the Einstein tensor must be zero for the Einstein theory. Setting Eqs. 3 to 5 equal to zero gives the following equations for the Schwartzschild exterior solution, which holds for ($r >= r_s$):

$$0 = e^{-\lambda}\,\{ -(\lambda'/r) + (1/r^2) \} - (1/r^2) \tag{12}$$

$$0 = e^{-\lambda} \{ (\nu'/r) + (1/r^2) \} - (1/r^2) \quad (13)$$

$$0 = e^{-\lambda} \{ ½ (\nu'/r) - ½ (\lambda'/r) + ½ \nu'' - ¼ \lambda' \nu' + ¼ (\nu')^2 \} \quad (14)$$

Results of Schwartzschild analysis. Tolman [5] (pp. 243-247) shows how Eqs 9 to 14 are solved to obtain the Schwartzschild exterior and interior solutions. As shown in Tolman's Eqs. 96.4 to 96.6, these equations simplify to the following three simultaneous equations, which hold inside the star

$$8\pi p = e^{-\lambda} \{ (\nu'/r) + (1/r^2) \} - (1/r^2) \quad (15)$$

$$8\pi\rho = e^{-\lambda} \{ (\lambda'/r) - (1/r^2) \} + (1/r^2) \quad (16)$$

$$dp/dr = - ½ (\rho + p)\nu' \quad (17)$$

On pages 246-247, Tolman derived from Eqs. 15 to 17 the following formula for pressure inside the star:

$$p = A/B$$

$$A = 3m\{ \sqrt{[1 - 2(m/r_s)(r/r_s)^2]} - \sqrt{[1 - 2(m/r_s)]} \}$$

$$B = 4\pi r_s^3 \{ 3\sqrt{[1 - 2(m/r_s)]} - \sqrt{[1 - 2(m/r_s)(r/r_s)^2]} \} \quad (18)$$

At the surface of the star, where r is equal to r_s, this gives a pressure of zero, as it should be. However if $2m/r_s$ is greater than unity, the pressure becomes imaginary inside the star at small values of radius. Since this is a physically impossible result, the Schwartzchild analysis does not yield a solution for m/r_s greater than ½. Within the limits of the analysis (which are for $2m/r_s$ <=1), the Schwartzschild solution yields the metric tensor values in spherical coordinates that are shown in Table 2-1. These metric tensor elements were obtained from Tolman [5] on p. 204 (Eq. 82.9) and on p. 247 (Eqs. 96.9, 96.13).

Table 2-1: *Metric tensor values in spherical coordinates for the Schwartzschild exterior and interior solutions, which apply to a single star*

element	exterior	interior solution
g_{00}	$1 - 2(m/r)$	$¼\{3\sqrt{[1-2(m/r_s)]} - \sqrt{[1-2(m/r_s)(r/r_s)^2]}\}^2$
g_{11}	$-1/\{1-2(m/r)\}$	$-1/[1 - 2(m/r_s)(r/r_s)^2]$
g_{22}	$-r^2$	$-r^2$
g_{33}	$-(r \sin\theta)^2$	$-(r \sin\theta)^2$

By Eq. 18, the pressure p inside the star is infinite if B is zero, which occurs if

$$3\sqrt{[1 - 2(m/r_s)]} = \sqrt{[1 - 2(m/r_s)(r/r_s)^2]} \qquad (19)$$

Squaring this equation gives

$$9 - 18(m/r_s) = 1 - 2(m/r_s)(r/r_s)^2 \qquad (20)$$

Solving for $(r/r_s)^2$ gives

$$(r/r_s)^2 = 9 - [4/(m/r_s)] \qquad (21)$$

At the value of r given by Eq. 21 the pressure inside the star is infinite. When m/r_s is 4/9, the pressure is infinite at the center of the star. When m/r_s lies between 4/9 and ½, there is a spherical surface inside the star over which the pressure is infinite. As m/r_s increases from 4/9 to ½, this spherical surface of infinite pressure expands from the center of the star to the circumference. Since infinite pressure inside a star is physically impossible, it is generally assumed that a star cannot exist if the m/r ratio at the surface of a star exceeds 4/9.

2, The Black Hole and Event Horizon Concepts

Section 4 will show that the apparent relative speed of light in the radial direction is equal to

$$c_{ap}/c = \sqrt{[-g_{00}/g_{11}]} \quad \text{(Schwartzschild radial motion)} \qquad (22)$$

As will be shown in Section 4, this relation holds only in the radial direction for the Schwartzschild solution, but it holds in all directions for the Einstein isotropic solution and for the Yilmaz theory. For tangential motion (perpendicular to the radius), the c_{ap}/c ratio for the Schwartzschild solution is

$$c_{ap}/c = \sqrt{[g_{00}]} \text{(Schwartzschild tangential motion)} \qquad (23)$$

Equations 22, 23 both show that the speed of light is zero when g_{00} is zero.

If m/r_s is equal to ½, Table 1 shows that g_{00} should be zero at the surface of the star, and so the speed of light should be zero over the surface. Light presumably cannot escape from such a star, and so the star is called a *black hole*. A spherical surface over which the speed of light is zero is called an *event horizon*.

When m/r_s exceeds ½, the pressure inside the star becomes imaginary in the Schwartzschild solution. It was originally assumed that the Schwartzschild analysis does not apply under this condition. However later studies have led to the conclusion that when m/r_s exceeds ½ the star must collapse to form a singularity at its center of infinite mass density. Under this assumption, the interior solution no longer exists, and so the exterior solution can still apply. For such a star there should be an event-horizon spherical surface beyond the original radius of the star, which occurs at a radius r equal to 2m. Over this event-horizon the speed of light should be zero. Light presumably cannot escape from the volume inside this event-horizon sphere.

This event horizon, which is derived from the exterior Schwartzschild solution, falls at or outside the original radius of the star when the m/r_s ratio is equal to or greater than ½. Another event horizon is predicted by the interior Schwartzschild solution, and falls inside the star. This event horizon inside the star occurs if m/r_s is greater than 4/9 but less than ½. Table 2-1 shows that g_{00} for the interior solution is zero inside the star under the same conditions where the quantity B in Eq. 18 is zero, and so the pressure p inside the star is infinite.

At the value of r given by Eq. 21, g_{00} is zero and so the speed of light should be zero. The equation shows that when m/r_s is equal to 4/9, the speed of light should be zero at the center of the star and the pressure should be infinite. When m/r_s lies between 4/9 and ½, there should be an event-horizon spherical surface inside the star where the speed of light is zero and the pressure is infinite. As m/r_s increases from 4/9 to ½, this event-horizon surface should expand from the center of the star to the circumference.

A star cannot exist in a normal state if it has an event horizon surface inside the star, because electromagnetic fields cannot penetrate this boundary, and the pressure inside the star should be infinite over this surface. Therefore it is generally assumed that a star must collapse into a black hole if m/r_s exceeds 4/9. This writer takes the position that the *black hole* and *event horizon* concepts do not represent physical reality, because they are not displayed in the Yilmaz theory. They are merely symptoms of a mathematical weakness in the Einstein theory.

2.3 The Isotropic Solution

As shown by Tolman [5] (p. 245), a second solution for a single star was derived by assuming that the metric equation is isotropic, which requires that the values of g_{11}, g_{22}, g_{33} be equal in rectangular coordinates. The resultant solution is called the *isotropic solution*. The corresponding metric tensor elements in spherical coordinates have the form

$$g_{00} = e^{\nu} \; ; \; g_{11} = - e^{\mu} \; ; \; g_{22} = - r^2 g_{11} \; ; \; g_{33} = - (r \sin \theta)^2 g_{11} \qquad (24)$$

The variables μ and ν are assumed to vary only with radial distance r.

As shown by Tolman [5] (p. 242, Eq. 95.6), the following formulas for the non-zero elements of the corresponding Einstein tensor are calculated from these metric tensor values:

$$G_0^{\,0} = e^{-\mu} \{ \mu'' + (\mu'^2/4) + (2\mu'/r) \} \qquad (25)$$

$$G_1^{\,1} = e^{-\mu} \{ ¼ \, \mu'^2 + ½ \, \mu'\nu' + (\mu'/r) + (\nu'/r) \} \qquad (26)$$

$$G_2^{\,2} = G_3^{\,3} = e^{-\lambda} \{ ½ \, \mu'' + ½ \, \nu'' + ¼ \, \nu'^2 + (\mu'/2r) + (\nu'/2r) \} \qquad (27)$$

Just as for the Schwartzschild solution, these Einstein tensor values are set equal to zero to obtain equations for the *exterior isotropic solution*, and are set equal to the energy-momentum tensor values of Eq. 6 to obtain equations for the *interior isotropic solution*. Tolman (p. 244, Eq. 95.15) shows that this analysis reduces to the following three simultaneous equations:

$$8\pi p = e^{-\mu} \{ \tfrac{1}{4} \mu'^2 + \tfrac{1}{2} \mu'\nu' + (\mu'/r) + (\nu'/r) \} \tag{28}$$

$$8\pi\rho = - e^{-\mu} \{ \mu'' + \tfrac{1}{4} \mu'^2 + 2(\mu'/r) \} \tag{29}$$

$$dp/dr = - \tfrac{1}{2} (\rho + p)\nu' \tag{30}$$

Equation 30 is the same as Eq. 17 for the Schwartzschild solution. As explained by Tolman (p. 246), Eq. 30 has the following solution, where C is an unspecified constant:

$$\rho + p = C\, e^{-\nu/2} \tag{31}$$

However Eqs. 28, 29 cannot be reduced to an analytical expression for the pressure p, as was done with the Schwartzschild solution, and so this analysis for the isotropic solution has never been completed. Nevertheless one should be able to obtain a solution with a computer using numerical analysis.

As explained by Tolman [5] (p. 205, Eqs. 82.11, 82.12), formulas for the exterior isotropic solution have been derived from the formulas for the exterior Schwartzschild solution as follows. The metric equation for the exterior Schwartzschild solution in spherical coordinates is

$$ds^2 = [1 - 2(m/r)](d\tau)^2 - [1 - 2(m/r)]^{-1}(dr)^2 - r^2 (d\theta)^2 - (r \sin \theta)^2 (d\theta)^2 \tag{32}$$

The radial distance r in Eq. 32 is replaced by the following expression

$$r = \{ 1 + (m/2\underline{\mathbf{r}}) \}^2\, \underline{\mathbf{r}} \tag{33}$$

This yields the following equation, which is isotropic relative to the variable $\underline{\mathbf{r}}$:

$$ds^2 = g_{00}(d\tau)^2 + g_{11} \{ (d\underline{\mathbf{r}})^2 + \underline{\mathbf{r}}^2(d\theta)^2 + (\underline{\mathbf{r}} \sin \theta)^2 (d\psi)^2 \} \tag{34}$$

where g_{00}, g_{11} are equal to

$$g_{00} = \{1 - (m/2\underline{\mathbf{r}})\}^2/\{1 + (m/2\underline{\mathbf{r}})\}^2 \tag{35}$$

$$g_{11} = - \{1 + (m/2\underline{\mathbf{r}})\}^4 \tag{36}$$

When the isotropic solution is used, the variable $\underline{\mathbf{r}}$ is usually treated as the radial distance r. However, it is not clear what the radial distance variable means for this solution.

2.4 Implications of Isotropy

The isotropic solution of the Einstein theory and all solutions of the static Yilmaz theory have isotropic metric tensors. In rectangular coordinates, an isotropic metric tensor is diagonal and the values of g_{11}, g_{22}, g_{33} are equal. Hence the metric equation for an isotropic metric tensor has the following form in rectangular coordinates:

$$ds^2 = g_{00}(d\tau)^2 + g_{11} \{ (dx)^2 + (dy)^2 + (dz)^2 \} \qquad (37)$$

It can be shown that the differential coordinates for rectangular and spherical coordinates are related by.

$$(dx)^2 + (dy)^2 + (dz)^2 = (dr)^2 + r^2 (d\theta)^2 + (r \sin \theta)^2 (d\theta)^2 \qquad (38).$$

Substituting Eq. 38 into Eq. 37 gives the isotropic metric equation in spherical coordinates:

$$ds^2 = g_{00}(d\tau)^2 + g_{11} \{ (dr)^2 + r^2 (d\theta)^2 + (r \sin \theta)^2 (d\theta)^2 \}$$

$$= g_{00}(d\tau)^2 + g_{11} (dr)^2 + r^2 g_{11} (d\theta)^2 + (r \sin \theta)^2 g_{11} (d\theta)^2 \qquad (39)$$

Thus for an isotropic metric equation, the metric tensor elements g_{00}, g_{11} in spherical coordinates are the same as in rectangular coordinates, and the elements g_{22}, g_{33} in spherical coordinates are equal to:

$$g_{22} = r^2 g_{11} \ ; \quad g_{33} = (r \sin \theta)^2 g_{11} \qquad (40)$$

Let us replace the differential d by the incremental quantity Δ. Equation 37 shows that an isotropic metric equation in rectangular coordinates can be expressed as

$$(\Delta s)^2 = g_{00}(\Delta\tau)^2 + g_{11} (\Delta x_d)^2 \qquad (41)$$

where Δx_d is the total distance between two points, which is equal to

$$(\Delta x_d)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \qquad (42)$$

Document *1,7 Metric Equation* shows, in Table 3, that, when the metric equation is isotropic, the apparent speed of light and the apparent compression of distance are given by

$$c_{ap}/c = \sqrt{[-g_{00}/g_{11}]} \qquad (43)$$

$$\Delta x_{ap}/\Delta x = 1/\sqrt{[-g_{11}]} \qquad (44)$$

For an isotropic metric tensor, these equations apply to motion in any direction. The metric equation for the Schwartzschild solution is not isotropic, and it is diagonal only in spherical coordinates. In spherical coordinates this metric equation is

$$\Delta s^2 = g_{00}(\Delta\tau)^2 + g_{11}(\Delta r)^2 - r^2(\Delta\theta)^2 - (r \sin\theta)^2(\Delta\psi)^2 \qquad (45)$$

Consider incremental linear displacements in the θ and ψ directions, and denote these as Δx_θ and Δx_ψ. These linear displacements are related as follows to the angular displacements of θ and ψ:

$$\Delta x_\theta = r\,\Delta\theta \qquad (46)$$

$$\Delta x_\psi = r \sin\theta\,\Delta\psi \qquad (47)$$

Substituting Eqs. 46, 47 into Eq. 45 gives

$$\Delta s^2 = g_{00}(\Delta\tau)^2 + g_{11}(\Delta r)^2 - (\Delta x_\theta)^2 - (\Delta x_\psi)^2 \qquad (48)$$

The displacements Δx_θ, Δx_ψ are tangential linear displacements that are normal to the radial vector. Let us denote the total tangential displacement as Δx_t. This is related as follows to the displacements in the θ and ψ directions:

$$(\Delta x_t)^2 = (\Delta x_\theta)^2 + (\Delta x_\psi)^2 \qquad (49)$$

Substituting Eq. 49 into Eq. 48 gives:

$$\Delta s^2 = g_{00}(\Delta\tau)^2 + g_{11}(\Delta r)^2 - (\Delta x_t)^2 \qquad (50)$$

This is a convenient form of the metric equation for the Schwartzschild solution. From this we can obtain formulas for the speed of light and for spatial compression for motion in the radial direction by setting Δx_t equal to zero. We can obtain the corresponding formulas for motion in the tangential direction by setting Δr equal to zero

By applying these principles, the formulas for speed of light and spatial compression for the Schwartzschild solution can be calculated as follows for motion in radial and tangential directions. (The exterior Schwartzschild values of g_{00}, g_{11} in Table 2-1 were applied in these equations.)

Radial: $$c_{ap}/c = \sqrt{[-g_{00}/g_{11}]} = 1 - 2(m/r) \qquad (51)$$

Tangential: $$c_{ap}/c = \sqrt{[g_{00}]} = \sqrt{[1 - 2(m/r)]} \qquad (52)$$

Radial: $$\Delta r_{ap}/\Delta r = 1/\sqrt{[-g_{11}]} = \sqrt{[1 - 2(m/r)]} \qquad (53)$$

Tangential: $$\Delta x_{ap}/\Delta x = 1 \qquad (54)$$

For the isotropic solution, the effects of motions are the same in the radial and tangential directions. The formulas for the relative speed of light and for spatial compression are as follows for the isotropic solution. (The values for g_{00}, g_{11} were obtained from Eqs. 35, 36.)

$$c_{ap}/c = \sqrt{[-g_{00}/g_{11}]} = \{ 1 - (m/2r) \}/\{ 1 + (m/2r) \}^3 \qquad (55)$$

$$r_{ap}/\Delta r = 1/\sqrt{[-g_{11}]} = 1/\{ 1 + (m/2r) \}^2 \qquad (56)$$

5, Plots of Speed of Light and Spatial Compression

Figure 2-1 shows the plots of the relative speed of light for: (1) the isotropic Einstein solution, (2) the Schwartzschild solution in the radial direction, and (3) the Yilmaz solution. The Schwartzschild (radial) solution was obtained from Eq 51, the Einstein isotropic solution from Eq. 55, and the Yilmaz solution from Table 8-5 of Chapter 8 of *Believe* [1].

Figure 2-1 shows that the speed of light for the Einstein isotropic solution is very close to that for the Yilmaz solution. The isotropic solution is zero at m/r equal to 2, but it is not zero at the Schwartzschild singularity point at m/r equal to ½.

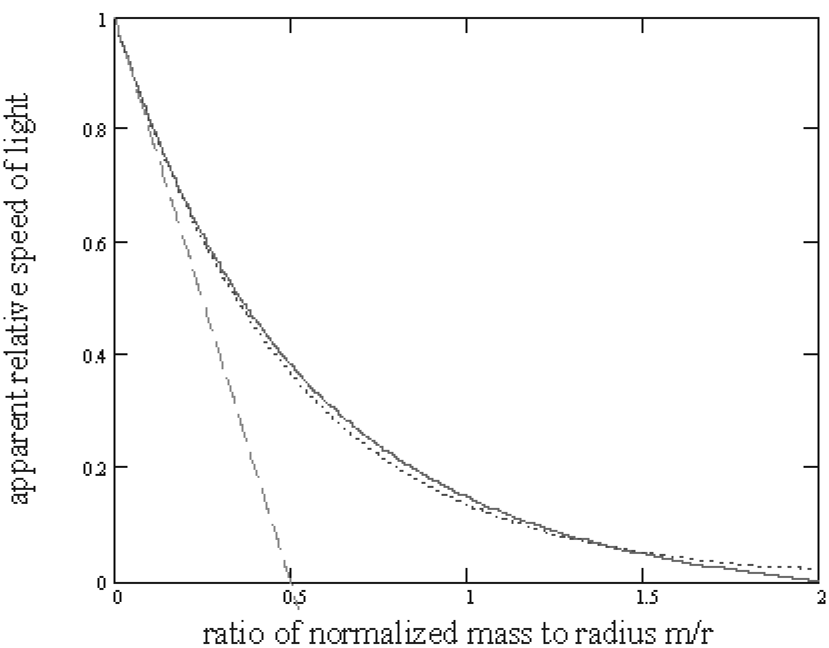

Figure 2-1: Apparent relative speed of light for Einstein isotropic solution (solid), for Yilmaz theory (short dashes), and for Einstein Schwartzschild solution (long dashes).

Figure 2-2 shows the speed of light for the Einstein Schwartzschild solution for radial motion (solid curve) and for tangential motion (dashed curve). These plots are radically different. It should be noted that the speed of light is different for these two directions even for weak gravitational fields like our solar system, where m/r is very much less than unity. It can be shown from Eqs. 51, 52 that the relative speed of light for the Schwartzschild solution is accurately approximately as follows in the weak gravitational fields of our solar system:

Radial: $c_{ap}/c \cong 1 - 2(m/r)$ (57)

Tangential: $c_{ap}/c \cong 1 - (m/r)$ (58)

Figure 2-3 compares the spatial compression caused by gravity for the Einstein isotropic solution (solid curve), the Yilmaz solution (short dashes), and for the Schwartzschild solution in the radial direction (long dashes). As shown by Eq. 54, the Schwartzschild solution does not exhibit spatial compression in the tangential direction. Figure 2-3 shows that the spatial compression for the isotropic solution is quite similar to that for the Yilmaz theory and it differs radically from that of the Schwartzschild solution.

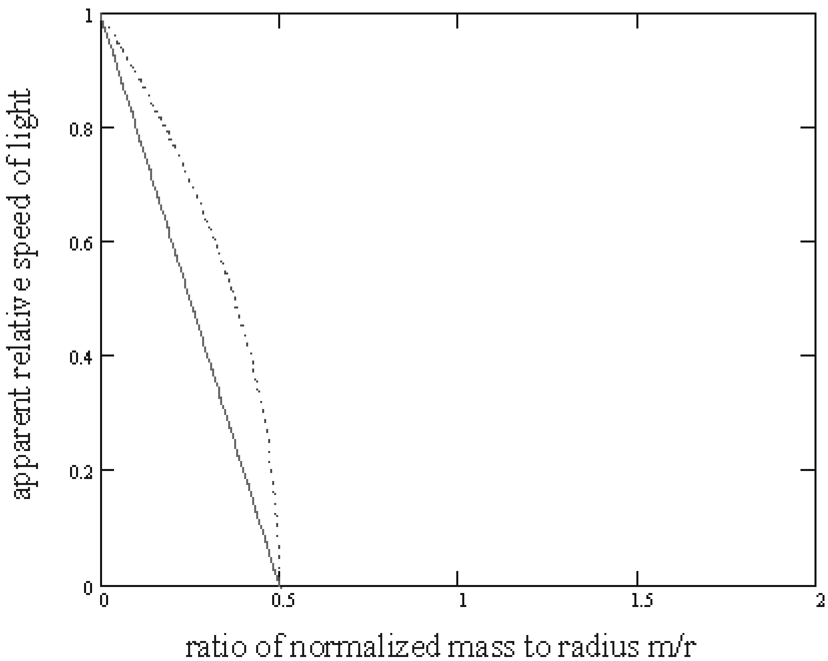

Figure 2-2: Apparent relative speed of light for Einstein Schwartzchild solution for radial motion (solid) and for tangential motion (dashes).

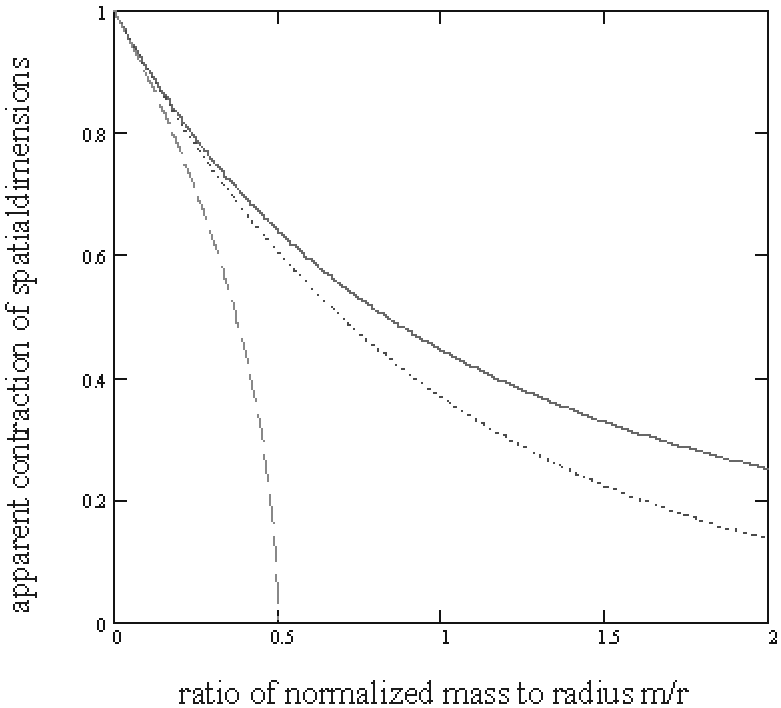

Figure 2-3: Apparent contraction of spatial dimensions for Einstein isotropic solution (solid), for Yilmaz theory (short dashes), and for Einstein Schwartzschild solution (long dashes).

References

[1] Adrian Bjornson, *A Universe that We Can Believe, Addison :Press, 2000, described in world-wide website Olduniverse.com* ISBN 09703231-0-7.

[5] Richard C. Tolman, *Relativity, Thermodynamics, and Cosmology,* Dover Publications (31 East 2nd St., Mineola, NY, 11501), 1987 (first published, 1934, Oxford, Clarendon Press) ISBN 0-486-65383-8.