5,3 Addendum Chapter 3
Aspects of Einstein and Yilmaz Relativity Theories

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This chapter discusses various aspects of the Einstein and Yilmaz theories. Section 1 analyzes the gravitational field equation for the static solution of the Yilmaz theory, and shows that it is exactly satisfied for the general static case. Section 2 discusses the Poisson equation and shows that the Einstein gravitational equation is closely related to the Poisson equation. The Poisson equation is used to calculate the gravitational potential for the Yilmaz theory when the mass density is spherically symmetric.

Section 3 discusses the Einstein pseudo-tensor for the gravitational field, which Einstein considered as representing the "energy components of the gravitational field". This pseudo tensor is related to the Yilmaz stress-energy tensor for the gravitational field, which is a true tensor. Section 4 discusses the concept of the covariant derivative and shows that the Einstein and Yilmaz gravitational field equations are based on the Einstein tensor $G_\mu{}^\nu$, rather than the Ricci tensor $R_\mu{}^\nu$, because the covariant derivative of the Einstein tensor is zero.

1, Validating Yilmaz Gravitational Field Equation for the Static Solution

A very important aspect of the Yilmaz theory of gravitation is that its gravitational field equation is automatically satisfied when the gravitational potential tensor (or the gravitational potential for the static solution), is appropriately designed. (5,5) Chapter 5 gives an elaborate analysis to prove that the gravitational field equation is always satisfied for the general time-varying Yilmaz theory. This section addresses the simpler problem of proving that the gravitational field equation is always satisfied for the static Yilmaz solution

The gravitational field equation for the Yilmaz theory is

$$G_\mu{}^\nu = R_\mu{}^\nu - ½\, \delta_\mu{}^\nu R = -2(\tau_\mu{}^\nu + t_\mu{}^\nu) \qquad (1)$$

The tensor $\tau_\mu{}^\nu$ is the stress-energy tensor for matter, and $t_\mu{}^\nu$ is the stress-energy tensor for the gravitational field. The static Yilmaz theory is implemented by calculating the gravitational potential ϕ. The gravitational potential ϕ at the point $\mathbf{x_p}$ is computed from

$$\phi(\mathbf{x_p}) = \Sigma\, \Delta m_k/|\mathbf{x_p} - \mathbf{x_k}| \qquad (2)$$

The vector $\mathbf{x_k}$ denotes the location of the mass element Δm_k, and $|\mathbf{x_p} - \mathbf{x_k}|$ is the absolute value of the distance from the mass element to the point $\mathbf{x_p}$ where the gravitational potential is calculated.

In rectangular coordinates the metric tensor is diagonal and has the following elements

$$g_{00} = e^{-2\phi} \;;\; g_{11} = g_{22} = g_{33} = -e^{2\phi} \qquad (3)$$

Chapter 5 derives the following formulas for the stress-energy tensors in the static Yilmaz theory:

$$t_\mu^{\ \nu} = -\partial_\mu\phi\, \partial^\nu\phi + ½\, \delta_\mu^{\ \nu}\, \Sigma_\lambda\, \partial_\lambda\phi\, \partial^\lambda\phi \qquad (4)$$

$$\tau_0^{\ 0} = -e^{-2\phi}\, \nabla^2\phi = -e^{-2\phi}\{\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2\} \qquad (5)$$

In the static Yilmaz theory, the stress-energy tensor for matter $\tau_\mu^{\ \nu}$ reduces to the single element $\tau_0^{\ 0}$ given by Eq. 3. All other elements of this tensor are zero.

Appendix B gives formulas derived by Prof.. Herbert Dingle for calculating the elements of the Einstein tensor $G_\mu^{\ \nu}$ when the metric tensor is diagonal. These Dingle formulas were applied to the general metric tensor values of the static Yilmaz solution given in Eq. 3 (with derivatives relative to time set to zero). This yielded the following general formulas for the Einstein tensor of the static Yilmaz solution in rectangular coordinates:

$$G_\mu^{\ \nu} = 0 \qquad \text{for } \mu \neq \nu,\ \mu \text{ or } \nu = 0 \qquad (6)$$

$$G_j^{\ k} = -2\, e^{-2\phi}\, \partial_j\phi\, \partial_k\phi \qquad \text{for } j \neq k \qquad (7)$$

$$G_0^{\ 0} = e^{-2\phi}\, \Sigma_j\, \{(\partial_j\phi)^2 + 2\, \partial_j^{\ 2}\phi\} \qquad (8)$$

$$G_1^{\ 1} = -e^{-2\phi}\, \{(\partial_1\phi)^2 - (\partial_2\phi)^2 - (\partial_3\phi)^2\} \qquad (9)$$

$$G_2^{\ 2} = -e^{-2\phi}\, \{(\partial_2\phi)^2 - (\partial_1\phi)^2 - (\partial_3\phi)^2\} \qquad (10)$$

$$G_3^{\ 3} = -e^{-2\phi}\, \{(\partial_3\phi)^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2\} \qquad (11)$$

The calculus computations for deriving these elements of the Einstein tensor are very difficult and tedious. A skilled and patient mathematician may require two weeks to calculate these formulas without error, even with the great assistance of the Dingle formulas.

From Eqs. 4, 5 one can derive the following equations for the stress-energy tensor elements in rectangular coordinates:

$$\tau_0^{\ 0} = -e^{-2\phi}\, \Sigma_k\, \partial_k^{\ 2}\phi \qquad (12)$$

$$t_0^{\ 0} = -½\, e^{-2\phi}\, \Sigma_k\, (\partial_k\phi)^2 \qquad (13)$$

$$t_j^{\ j} = e^{-2\phi}\{(\partial_j\phi)^2 - ½\, \Sigma_k\, (\partial_k\phi)^2\} \qquad (j = 1, 2, 3) \qquad (14)$$

$$t_j^{\ k} = e^{-2\phi}\, \partial_j\phi\, \partial_k\phi \qquad (j \neq k;\ j, k = 1, 2, 3) \qquad (15)$$

$$t_\mu^{\ \nu} = 0 \qquad (\nu \neq \mu;\ \nu \text{ or } \mu = 0) \qquad (16)$$

The summations are implemented over the three values (1, 2, 3) of the index k.

Equation 12 shows that the stress-energy tensor for matter $\tau_\mu{}^\nu$ has only one nonzero element $\tau_0{}^0$, which is equal to

$$\tau_0{}^0 = - e^{-2\phi} \{\partial_1{}^2\phi + \partial_2{}^2\phi + \partial_3{}^2\phi\} \quad (17)$$

Equations 13 to 16 yield the following formulas for the elements of the stress-energy tensor for the gravitational field $t_\mu{}^\nu$

$$t_0{}^0 = - ½ e^{-2\phi} \{ (\partial_1\phi)^2 + (\partial_2\phi)^2 + (\partial_3\phi)^2 \} \quad (18)$$

$$t_1{}^1 = ½ e^{-2\phi} \{ (\partial_1\phi)^2 - (\partial_2\phi)^2 - (\partial_3\phi)^2 \} \quad (19)$$

$$t_2{}^2 = ½ e^{-2\phi} \{ (\partial_2\phi)^2 - (\partial_1\phi)^2 - (\partial_3\phi)^2 \} \quad (20)$$

$$t_3{}^3 = ½ e^{-2\phi} \{ (\partial_3\phi)^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2 \} \quad (21)$$

$$t_j{}^k = e^{-2\phi} \partial_j\phi\, \partial_k\phi \qquad (j, k = 1, 2, 3; k \neq j) \quad (22)$$

$$t_\mu{}^\nu = 0 \qquad (\nu \text{ or } \mu = 0; \nu \neq \mu) \quad (23)$$

Based on the above values of the stress-energy tensors, the expressions for $-2(\tau_\mu{}^\nu + t_\mu{}^\nu)$ are as follows:

$$- 2(\tau_\mu{}^\nu + t_\mu{}^\nu) = 0 \qquad (\nu \neq \mu; \nu \text{ or } \mu = 0) \quad (24)$$

$$- 2(\tau_j{}^k + t_j{}^k) = - 2e^{-2\phi} \partial_j\phi\, \partial_k\phi \qquad (j \neq k; j, k = 1, 2, 3) \quad (25)$$

$$- 2(\tau_0{}^0 + t_0{}^0) = e^{-2\phi} \Sigma_k \{ 2\partial_k{}^2\phi + (\partial_k\phi)^2 \} \quad (26)$$

$$- 2(\tau_1{}^1 + t_1{}^1) = - e^{-2\phi} \{ (\partial_1\phi)^2 - (\partial_2\phi)^2 - (\partial_3\phi)^2 \} \quad (27)$$

$$- 2(\tau_2{}^2 + t_2{}^2) = - e^{-2\phi} \{ (\partial_2\phi)^2 - (\partial_1\phi)^2 - (\partial_3\phi)^2 \} \quad (28)$$

$$- 2(\tau_3{}^3 + t_3{}^3) = - e^{-2\phi} \{ (\partial_3\phi)^2 - (\partial_1\phi)^2 - (\partial_2\phi)^2 \} \quad (29)$$

In Eqs. 24 to 29, the values for $\tau_\mu{}^\nu$ are all zero, except for the value for $\tau_0{}^0$ in Eq. 26, which is obtained from Eq. 17.

Comparing Eqs. 24 to 29 with Eqs. 6 to 11 shows that $G_\mu{}^\nu$ is exactly equal to $-2(\tau_\mu{}^\nu + t_\mu{}^\nu)$ for all elements of these tensors. This proves that the Yilmaz gravitational field equation is exactly satisfied for all static solutions of the Yilmaz theory.

Conclusion. An examination of the formulas required to implement this calculation will show that even for this static case, where the metric tensor is diagonal, the equations for calculating the elements of the Einstein tensor $G_\mu{}^\nu$ from the metric tensor $g_{\mu\nu}$ are very complicated. These calculations are radically different from those for calculating the elements of the stress-energy tensors $\tau_\mu{}^\nu$, $t_\mu{}^\nu$. One would never dream when performing these difficult computations that both paths lead to the same result; that $G_\mu{}^\nu$ is exactly equal to the expression -2($\tau_\mu{}^\nu + t_\mu{}^\nu$) for all elements of the tensors.

This result is remarkable! The analysis shows that the gravitational field equation of the Yilmaz theory is exactly satisfied for all cases of the static theory. This proves that the Yilmaz theory has a very solid and profound mathematical foundation.

2, The Poisson Equation

Relation between Gravitational Field Equation and Poisson Equation

Newton's laws of mechanics are characterized by equations that describe the forces exerted on bodies, and the resultant velocities and accelerations of the bodies. About 1800 the French mathematician Poisson developed a field theory approach to Newton's laws by expressing them in terms of the gravitational potential. Einstein extended this gravitational field theory concept in developing his gravitational field equation.

The Poisson equation deals with a *gravitational potential* ϕ' that is related as follows to the relativistic *gravitational potential* ϕ of the Yilmaz theory:

$$\phi' = (c^2/G)\,\phi \qquad (30)$$

This shows that the equations for the Poisson gravitational potential ϕ' are obtained from those for the Yilmaz gravitational potential ϕ by replacing the relativistic mass m by the true mass M. Hence, the discussion of the Yilmaz theory in Appendix B of *Believe* [1] shows that the Poisson gravitational potential ϕ' generated at a point (p) by a collection of masses M_k is equal to

$$\phi' = \Sigma\, M_k/|r_{pk}| \qquad (31)$$

This represents a summation over the masses. The quantity $|r_{pk}|$ is the absolute value of the distance between the point (p) and the center of gravity of mass M_k. If a test mass M_t that is free to move is placed in this gravitational field, it can be shown (by applying Newton's laws of mechanics) that the acceleration of the test mass in the x-direction is equal to

$$A_x = -\,\partial\phi'/\partial x \qquad (32)$$

Thus, the acceleration in the x-direction is equal to the negative of the partial derivative of the gravitational potential ψ in the x-direction. Equivalent equations hold for all three spatial directions, x, y, z. The gravitational force applied to the test mass in the x-direction is obtained by multiplying this acceleration by the mass M_t of the test particle to obtain:

$$f_x = M_t\, A_x = -\,M_t\,(\partial\phi'/\,\partial x) \qquad (33)$$

By applying calculus to Eq. 3-31, it can be shown that the following formula holds at all points that are exterior to the bodies:

$$(\partial^2\phi'/\partial x^2) + (\partial^2\phi'/\partial y^2) + (\partial^2\phi'/\partial z^2) = 0 \qquad (34)$$

The expression $(\partial^2\phi'/\partial x^2)$ is the second partial derivative of ϕ' relative to x. This is obtained by calculating the partial derivative of ϕ' relative to x, and then calculating the partial derivative of this result relative to x. When we consider a continuous medium of matter of mass density ρ, Eq. 3-34 becomes

$$(\partial^2\phi'/\partial x^2) + (\partial^2\phi'/\partial y^2) + (\partial^2\phi'/\partial z^2) = - 4\pi\rho \qquad (35)$$

This is *Poisson's* equation. The form of this equation can be simplified by defining the symbol ∇^2 (called the Laplacian) which represents the following operation in rectangular coordinates:

$$\nabla^2 = (\partial^2/\partial x^2) + (\partial^2/\partial y^2) + (\partial^2/\partial z^2) \qquad (36)$$

Using this shorthand notation, the Poisson formula of Eq. 35 becomes

$$- \nabla^2\phi = 4\pi\rho \qquad (37)$$

In a region of space where there is no mass (a vacuum), the Poisson equation reduces to:

$$\nabla^2\phi = 0 \qquad (38)$$

We should compare Eq. 37 with the following formula for the Einstein gravitational field equation:

$$- ½\, G_\mu{}^\nu = 4\pi\, T_\mu{}^\nu \qquad (39)$$

For the Schwartzschild solution the value for $T_0{}^0$ is the mass density ρ. Hence for $\mu = \nu = 0$, Eq. 39 gives

$$- ½\, G_0{}^0 = 4\pi\rho \qquad (40)$$

This is similar to the Poisson formula of Eq. 37. Einstein designed his general theory of relativity to approximate the Poisson equation under weak gravitational fields. This assured that general relativity theory would approximately satisfy Newton's laws of mechanics within our solar system.

For the Yilmaz theory, the stress energy tensor for matter $\tau_\mu{}^\nu$ has only one nonzero element, which is

$$\tau_0{}^0 = - e^{-2\phi} \{\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 + \partial^2\phi/\partial z^2\} \qquad (41)$$

This was obtained from Eq. 5. In accordance with Eq. 36, this can be expressed as

$$\tau_0^{\ 0} = - e^{-2\phi} \nabla^2\phi \tag{42}$$

This illustrates the parallelism between the Poisson and Yilmaz formulas.

Use of Poisson Equation in Calculating Gravitational Potential for Yilmaz theory

This section applies the Poisson equation to calculate the gravitational potential inside a medium having constant mass density. Hildebrand [5] (p. 329) shows that the Laplacian in spherical coordinates is

$$\nabla^2\phi = (1/r^2)\partial_r\{r^2\partial_r\phi\} + (1/r^2\sin\theta)\partial_\theta\{\sin\theta\ \partial_\theta\phi\} + (1/r^2\sin^2\theta)\partial_\psi{}^2\phi \tag{43}$$

If the gravitational field is spherically symmetric, the derivatives of the gravitational potential ϕ relative to angles θ and ψ (denoted by ∂_θ, ∂_ψ) are zero. Hence Eq. 3-43 reduces to

for spherical symmetry: $$\nabla^2\phi = (1/r^2)\partial_r\{r^2\partial_r\phi\} \tag{44}$$

Applying this to Poisson's equation in Eq. 37 gives

$$\nabla^2\phi = (1/r^2)\ \partial\{r^2\partial\phi/\partial r\}/\partial r = - 4\pi\rho \tag{45}$$

The symbol $\partial_r F$ was replaced by $\partial F/\partial r$. Since this is considering only a single variable r, the partial derivatives can be replaced by simple derivatives, and Eq. 3-45 becomes

$$d\{r^2 d\phi/dr\} = - 4\pi\rho r^2\, dr \tag{46}$$

Integrating this gives

$$d\phi/dr = - (4\pi/r^2) \textstyle\int \rho r^2\, dr \tag{47}$$

If the density ρ is constant, this becomes

$$d\phi/dr = - (4\pi\rho/3)\ r \tag{48}$$

This yields

$$\phi = - (4\pi\rho/3) \textstyle\int r\, dr \tag{49}$$

Implementing the integration gives

$$\phi = - (2\pi\rho/3)r^2 + \text{constant} \tag{50}$$

If the density is not constant, one must implement the following integration:

$$\phi \;=\; -\,4\pi \int (dr/r^2) \int \rho(r)\, r^2\, dr \qquad (51)$$

3, The Einstein Pseudo Tensor for the Gravitational Field

The Yilmaz theory expresses the Einstein tensor as the sum of two stress-energy tensors, one describing the stress-energy of matter and the other describing the stress-energy of the gravitational field. One might wonder why Einstein did not include in his gravitational field equation a tensor to characterize the gravitational field. The answer is that he was unable to derive the appropriate tensor.

Einstein found a pseudo-tensor, which he called the “energy components of the gravitational field”. Einstein denoted his pseudo-tensor $t_\mu{}^\nu$, but to distinguish it from the true stress-energy tensor $t_\mu{}^\nu$ of the Yilmaz theory, we denote this Einstein pseudo-tensor with bold font and strike-through as $\mathbf{ŧ}_\mu{}^\nu$. The formula given by Pauli [6] to define this Einstein pseudo-tensor was given in Eq. G-1.

Pauli [6] (p. 162) states that Einstein considered the components of his pseudo-tensor $\mathbf{ŧ}_\mu{}^\nu$ to be equivalent in certain respects to the *energy-momentum components* $T_\mu{}^\nu$ *of matter*, which is a true tensor. In Paragraph 61 (p. 176), Pauli shows in his Eq. 447 that Einstein derived the following energy conservation law for a closed system in general relativity:

$$J_\mu \;=\; \Sigma\,(\sqrt{[-g]}\,T_\mu{}^0)\,\Delta v \;+\; \Sigma\,\mathbf{ŧ}_\mu{}^0\,\Delta v \;=\; \text{constant} \qquad (52)$$

We have replaced the calculus integral sign of the Pauli equation with an equivalent summation Σ sign, and have split the summation into two parts. The quantity Δv represents an increment of volume.

The expression ($\sqrt{[-g]}\,T_\mu{}^\nu$) is the *tensor density* of the energy-momentum tensor $T_\mu{}^\nu$. Multiplying the *tensor density* of the energy-momentum tensor $T_\mu{}^0$ by the increment of volume Δv should give the momentum and energy of matter within the volume increment Δv. Hence the summation given by the first term of Eq 31 should represent the total momentum and energy of matter.

Einstein considered J_μ in Eq. 52 to be the total momentum and energy of the closed system. Since $T_\mu{}^\nu$ characterizes the momentum and energy of matter, then $\mathbf{ŧ}_\mu{}^\nu$ would appear “at first sight” to be the *energy components of the gravitational field*. However on page 176 Pauli objects to this conclusion with the following comments

> "On closer inspection, however, great difficulties become apparent, which oppose this first-sight point of view. In the final analysis, they are due to the fact that the $\mathbf{ŧ}_\mu{}^\nu$ variables do not form a tensor. Since these quantities do not depend on the derivatives of $g_{\mu\nu}$ higher than the first, we can conclude immediately that they can be made to vanish at an arbitrarily prescribed world point for a suitable choice of the coordinate system.

"But we can go still further: Schrodinger found that all of the energy components vanish identically for the field of a point mass, which represents, at the same time, the field outside a liquid sphere. This result can also be extended to the case of the field of a charged sphere. On the other hand, Bauer showed that by simply introducing polar coordinates in the Euclidean line element of special relativity, the energy components are found to have values different from zero. In fact, the total energy becomes infinite ! Also, the $\mathfrak{t}_\mu^{\nu}$ variables are certainly not symmetrical, and the energy density $-\mathfrak{t}_0^{0}$ is not everywhere positive. In the earlier field theories of gravitation, it was the sign of the energy density of the gravitational field that always led to difficulties."

Pauli goes on to discuss the issue further, along with counter arguments that were presented by Einstein. However he does not refute the basic objections that are given here.

Yilmaz has added new insight to this historical enigma. As shown in Eq. 52, the Einstein pseudo-tensor $\mathfrak{t}_\mu^{\nu}$ is related as follows to the Yilmaz true tensor t_μ^{ν}:

$$4\pi\mathfrak{t}_\mu^{\nu} = - t_\mu^{\nu} + z_\mu^{\nu} \qquad (53)$$

The variable t_μ^{ν} is the Yilmaz stress-energy tensor for the gravitational field, which is a true tensor. The variable $\mathfrak{t}_\mu^{\nu}$ is the Einstein gravitational field pseudo-tensor, which "at first sight" seemed to characterize the energy components of the gravitational field. However $4\pi\mathfrak{t}_\mu^{\nu}$ is equal to the sum ($- t_\mu^{\nu} + z_\mu^{\nu}$) and so is corrupted by the non-tensor z_μ^{ν}. This non-tensor z_μ^{ν} keeps the Einstein variable $\mathfrak{t}_\mu^{\nu}$ from being a true tensor, and prohibited Einstein from using $\mathfrak{t}_\mu^{\nu}$ in his gravitational field equation.

When a true tensor is translated from one coordinate system to another, its elements are converted in accordance with the general tensor formula given in Eq. C-2 of Appendix C (or equivalent formulas for covariant and mixed tensors). The Einstein pseudo-tensor $\mathfrak{t}_\mu^{\nu}$ does not translate according to this formula and so is not a true tensor.

Yilmaz distilled the true tensor portion from the Einstein pseudo-tensor $\mathfrak{t}_\mu^{\nu}$, and thereby derived a true tensor t_μ^{ν} to characterize the energy and stress components of the gravitational field.

4, Form of the Gravitational Field Equation

Implication of Covariant Derivative in Gravitational Field Equation

One may wonder why the Einstein gravitational field equation uses the expression (R_μ^{ν} - ½ g_μ^{ν} R), which is the Einstein tensor G_μ^{ν}, rather than the Ricci tensor R_μ^{ν}. The answer relates to the covariant derivative, which is explained in Appendix J of *Believe* [1]. The *covariant derivative* is a generalization of the *partial derivative* concept that accounts for the curvature of space. The covariant derivative of a vector or a tensor describes the true change of the vector or tensor in curved space.

A fundamental theorem of tensor analysis, called the *Bianchi identity*, states that the covariant derivative of the Ricci tensor $R_\mu{}^\nu$ is always equal to the covariant derivative of ½ $g_\mu{}^\nu R$. Consequently the covariant derivative of the combined expression ($R_\mu{}^\nu$ - ½$g_\mu{}^\nu R$) is identically zero. This combined expression is the Einstein tensor $G_\mu{}^\nu$. Thus the Bianchi identity states that the covariant derivative of the Einstein tensor $G_\mu{}^\nu$ is identically zero, but the covariant derivative of the Ricci tensor $R_\mu{}^\nu$ is not.

The Einstein gravitational field equation in mixed form is

$$G_\mu{}^\nu = R_\mu{}^\nu - ½\, \delta_\mu{}^\nu R = -8\pi\, T_\mu{}^\nu \qquad (54)$$

Since the covariant derivative of $G_\mu{}^\nu$ is zero, the Einstein gravitational field equation requires that the covariant derivative of the energy-momentum tensor $T_\mu{}^\nu$ must be zero.

As explained by Landau and Lifshitz [7] (p. 280), in a flat gravity-free region of space, the conservation of energy and momentum of matter is achieved by setting the sum of the partial derivatives of the energy-momentum tensor $T^{\mu\nu}$ equal to zero:

$$\Sigma_\nu\, \partial T^{\mu\nu}/\partial x^\nu = 0 \qquad (55)$$

When one generalizes this concept to account for gravity, it is often claimed that conservation of energy and momentum of matter is achieved by setting the covariant derivative of $T^{\mu\nu}$ equal to zero:

$$\Sigma_\nu\, D_\nu(T_\mu{}^\nu) = 0 \qquad (56)$$

This condition is automatically satisfied by the Einstein gravitational field equation. Although it is often claimed that this property results in the conservation of energy and momentum of matter, Landau and Lifshitz [7] (p. 290) refutes this concept. They state that "this equation does not generally express any conservation law whatever". They point out that conservation of energy and momentum of matter is achieved only if the following derivative relation is satisfied:

$$\Sigma_\nu\, \partial\{\sqrt{[-g]}\, T_\mu{}^\nu\}/\partial x^\nu = 0 \qquad (57)$$

When this relation is satisfied the following integral is conserved:

$$\int \{\sqrt{[-g]}\, T_\mu{}^\nu\}\, dS \text{ is conserved} \qquad (58)$$

The variable $\{\sqrt{[-g]}\, T_\mu{}^\nu\}$ is the tensor density of the energy-momentum tensor $T_\mu{}^\nu$.

This book by the Russian authors Landau and Lifshitz, which is entitled, *The Classical Theory of Fields* [7], is a highly respected authoritative text on field theory and relativity theory. Its claims should not be taken lightly.

This discussion has shown that in the Einstein theory the Bianchi identity assures that the covariant derivative of the energy-momentum tensor is zero. It is commonly believed that this property guarantees conservation of energy and momentum of matter, but this is not true. In fact, as explained in Section 5.9 of Chapter 5, the Einstein theory requirement that the covariant derivative of the energy-momentum tensor must be zero conflicts with Eq. 3-61, which must be satisfied if the energy and momentum of matter is to be conserved. Because of this conflict, the equations of the Einstein theory are over-constrained and so can result in contradictory solutions.

In the Yilmaz gravitational field equation, the Einstein tensor is proportional to ($\tau_\mu{}^\nu$ + $t_\mu{}^\nu$), which is the sum of the stress-energy tensors for matter and the gravitational field. Hence For the Yilmaz theory the Bianchi identity requires that the covariant derivative of the sum ($\tau_\mu{}^\nu$ + $t_\mu{}^\nu$) must be zero. This requirement does not conflict with Eq. 3-61, which is expressed as follows in terms of the symbolism of the Yilmaz theory

$$\Sigma_\nu\, \partial\{\sqrt{[-g]}\tau_\mu{}^\nu\}/\partial x^\nu = 0 \qquad (59)$$

Principles Underlying Forms for Einstein and Yilmaz Gravitational Field Equations

This discussion has explained why the gravitational field equations of the Einstein theory is based on the Einstein tensor rather than the Ricci tensor. The analysis of the general time-varying Yilmaz theory in Chapter 5 explains why the Yilmaz theory has its form. Both theories use the Einstein tensor instead of the Ricci tensor because the covariant derivative of the Einstein tensor is identically zero.

The Einstein tensor $G_\mu{}^\nu$, which is equal to ($R_\mu{}^\nu$ - ½$g_\mu{}^\nu$R), is the foundation for the gravitational field equations of the Einstein and Yilmaz theories because it has the following properties:

(1) The covariant derivative of the Einstein tensor is zero.

(2) The Einstein tensor $G_{\mu\nu}$ is a second-order tensor, which means that it has two indices and 4x4 or 16 elements;

(3) The Einstein tensor includes first-order and second-order partial derivatives of the metric tensor, but no higher-order derivatives.

The Einstein tensor is the only curvature tensor with these three properties that can be derived from the metric tensor. Requirement (2) must be satisfied in order for general relativity to reduce to special relativity in the absence of a gravitational field. Requirement (3) must be satisfied in order for general relativity to approximate Newtonian theory in a weak gravitational field.

The Ricci tensor $R_{\mu\nu}$ is derived from the more general Riemann-Christoffel tensor, a physical discussion of which was given in Chapter 9 of *Believe* [1]. The Riemann-Christoffel tensor is denoted $R_{\mu\nu\sigma}{}^\tau$. Since this has four indices, it is a fourth-order tensor and so has 4x4x4x4 elements, or 256 elements. As shown by Tolman [8] (p. 484, Eq. 19), the formula for the Riemann-Christoffel tensor is

$$R_{\mu\nu\sigma}{}^{\tau} = \Sigma_\alpha (\Gamma_{\mu\sigma}{}^{\alpha}\Gamma_{\alpha\nu}{}^{\tau} - \Gamma_{\mu\nu}{}^{\alpha}\Gamma_{\alpha\sigma}{}^{\tau}) + \partial_\nu\Gamma_{\mu\sigma}{}^{\tau} - \partial_\sigma\Gamma_{\mu\nu}{}^{\tau} \quad (60)$$

To obtain the Ricci tensor $R_{\mu\nu}$ from this, the index τ is replaced by the index σ. Since the index σ is now repeated, the expression is summed over the four values (0, 1, 2, 3) of the index σ. The resultant formula for the Ricci tensor is

$$R_{\mu\nu} = \Sigma_\sigma \{\Sigma_\alpha (\Gamma_{\mu\sigma}{}^{\alpha}\Gamma_{\alpha\nu}{}^{\sigma} - \Gamma_{\mu\nu}{}^{\alpha}\Gamma_{\alpha\sigma}{}^{\sigma}) + \partial_\nu\Gamma_{\mu\sigma}{}^{\sigma} - \partial_\sigma\Gamma_{\mu\nu}{}^{\sigma} \} \quad (61)$$

Since the Ricci tensor $R_{\mu\nu}$ has two indices, it has 4x4 elements, or 16 elements.

As shown by Tolman [8] (p. 494, Eq. 18), the general Christoffel symbol formula is:

$$\Gamma_{\mu\nu}{}^{\alpha} = ½ \; \Sigma_\beta \; g^{\alpha\beta}\{ \partial_\nu g_{\mu\beta} + \partial_\mu g_{\nu\beta} + \partial_\beta g_{\mu\nu} \} \quad (62)$$

Since the Christoffel symbol has 3 indices, it has 4x4x4 elements, or 64 elements. The Christoffel symbol is not a tensor, and so does not transform from one coordinate system to another in accordance with the tensor transformation formula, which are discussed in Appendix C. However the Riemann-Christoffel tensor and the Ricci tensor, which are calculated from the Christoffel symbol, are true tensors.

As was explained in Chapter 9 of *Believe* [1], the Riemann-Christoffel tensor $R_{\mu\nu\sigma}{}^{\tau}$ describes the curvature of space. This tensor cannot be used directly in the gravitational field equation because it is a fourth-order tensor. The Riemann-Christoffel tensor is reduced to form the Ricci tensor $R_{\mu\nu}$, which is a second-order tensor. The Ricci tensor cannot be used directly in the gravitational field equation because its covariant derivative is not zero. The Einstein tensor $G_\mu{}^\nu$, which is equal to $(R_\mu{}^\nu - ½\, g_\mu{}^\nu R)$, is formed from the Ricci tensor to obtain a second-order tensor with a covariant derivative that is zero.

5, Energy-Momentum Tensors for Einstein and Yilmaz Theories

The Einstein and Yilmaz theories both have energy-momentum tensors. The Yilmaz theory calls its "energy-momentum tensor" the "stress-energy tensor for matter", but this is merely a terminology convenience that allows this tensor to be related conveniently to the Yilmaz "stress-energy tensor for the gravitational field". The Yilmaz theory uses the symbol $\tau_\mu{}^\nu$ for this tensor, which is equal to $4\pi T_\mu{}^\nu$, where $T_\mu{}^\nu$ is the energy-momentum tensor generally used for the Einstein theory. The following are the gravitational field equations for the two theories expressed in terms of $\tau_\mu{}^\nu$:

Einstein: $$G_\mu{}^\nu = - 2\tau_\mu{}^\nu \quad (63)$$

Yilmaz: $$G_\mu{}^\nu = - 2(\tau_\mu{}^\nu + t_\mu{}^\nu) \quad (64)$$

One might assume that the energy momentum tensor τ_μ^ν should be the same for the two theories. However, the following discussion shows that these two tensors are generally different.

Conservation of matter-plus-energy is a very important condition that should be satisfied by a theory of relativity. Matter can be converted into energy, and vice versa, in accordance with the Einstein formula ($E = mc^2$), but the total can neither be created nor destroyed. Landau and Lifshitz [7] (p. 280) demonstrate that to achieve conservation of matter-plus-energy requires that the following condition be satisfied:

For conservation of matter-plus energy:

$$\Sigma_\nu \ \partial_\nu\{\sqrt{[-g]}\tau_\mu^\nu\} = 0 \qquad (65)$$

As was stated earlier in Section 5, the Freud identity specifies that this condition is always satisfied by the Yilmaz theory. Hence the Yilmaz theory always achieves conservation of matter-plus-energy.

The Bianchi identity specifies that the covariant derivative of the Einstein tensor G_μ^ν is identically zero, as shown by

Bianchi identity: $$\Sigma_\nu \ D_\nu G_\mu^\nu \equiv 0 \qquad (66)$$

The symbol D_ν represents the covariant derivative, which is explained in Appendix J of *Believe* [1]. In the Einstein theory, the energy momentum tensor is proportional to the Einstein tensor. Hence the covariant derivative of the energy-momentum tensor must be zero, as shown by

Einstein theory: $$\Sigma_\nu \ D_\nu \tau_\mu^\nu = 0 \qquad (67)$$

It is commonly believed that satisfying this condition of Eq. 67 assures conservation of matter-plus-energy in the Einstein theory. However, Landau and Lifshitz [7] (p. 280) demonstrate that this belief is definitely not true. It can be shown that this condition of Eq. 67 merely achieves conservation of rest mass in the Einstein theory. The constraint on the energy momentum tensor specified by Eq. 67 conflicts with the conservation requirement of Eq. 65. Consequently, the Einstein theory generally does not achieve conservation of matter-plus-energy.

This discussion has shown that the energy-momentum tensor for the Yilmaz theory always achieves conservation of matter-plus-energy, but the energy-momentum tensor for the Einstein theory generally does not. Consequently the energy-momentum tensors are usually different in the Einstein and Yilmaz theories.

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