5,A Addendum Appendix A
Equations for Blackbody Radiator

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This appendix analyzes the standard spectral data for a blackbody radiator and derives several convenient relationships from this, including an expression for photon rate. The results are applied to calculate the photon rate for the cosmic microwave radiation received by the COBE satellite, and the photon rate that is radiated by the sun.

1, Handbook Data for Blackbody

Reference [5], page 6-153, gives the following equations describing an ideal blackbody radiator, where K_X is an unspecified constant, P is power, A is surface area, T is temperature in degrees Kelvin, and λ is wavelength:

$$dP/d\lambda \;=\; K_x/\lambda^5(\exp[K_1/\lambda T] - 1) \qquad (1)$$

$$K_1 \;=\; 1.438 \text{ cm-°K} \qquad (2)$$

$$P/A \;=\; 5.679\text{x}10^{-12}\, T^4 \;(\text{watt/cm}^2)/°K^4 \qquad (3)$$

$$\text{Max}[d(P/A)/d\lambda] \;=\; 1.290\text{x}10^{-11}\, T^5 (\text{watt/cm}^2)/\text{cm-°K}^5 \qquad (4)$$

In Eq. 4, the unit μm (which means 10^{-6} meter) was replaced by 10^{-4} cm. Dividing Eq. 4 by Eq. 3 gives

$$\text{Max}[dP/d\lambda] \;=\; 2.272\, T\, P \;(\text{cm-°K})^{-1} \qquad (5)$$

From Ref [5], page 6-154, the following were derived by interpolating between the data in the table:

$$\lambda_m T \;=\; 0.290 \text{ cm-°K} \qquad (6)$$

$$\lambda_h T \;=\; 0.4107 \text{ cm-°K} \qquad (7)$$

$$[dP/d\lambda] \text{ at } \lambda_h \;=\; 0.772 \text{ Max}[dP/d\lambda] \qquad (8)$$

where λ_m is the wavelength of maximum $dP/d\lambda$ and λ_h is the wavelength for half of the power integral. Half of the integral of the power spectrum lies below the wavelength λ_h and half lies above it. Substituting Eq. 5 into Eq. 8 gives

$$[dP/d\lambda] \text{ at } \lambda_h = 1.754 \; T\,P\,(\text{cm-°K})^{-1} \qquad (9)$$

2, Spectrum in Terms of Normalized Frequency

By combining the above information, the spectrum of Eq. 1 can be expressed as

$$dP/d\lambda = [56.39\, T\, P]/\{(\lambda/\lambda_h)^5(\exp[3.501(\lambda_h/\lambda)] - 1)\} \qquad (10)$$

This is equal to the value of Eq. 9 when $\lambda = \lambda_h$.

We need the power spectrum of a blackbody expressed in terms of frequency f, which is equal to c/λ. Rather than show the spectrum directly in terms of frequency f, it is often desirable to use the variable $(1/\lambda)$ as a normalized frequency, which is proportional to f. Since $d(1/\lambda) = -d\lambda/\lambda^2$, the spectrum relative to $(1/\lambda)$ is related as follows to the spectrum relative to λ

$$dP'/[d(1/\lambda)] = -\lambda^2\,[dP'/d\lambda] = \lambda^2\,[dP/d\lambda] \qquad (11)$$

The variable P' (for the $1/\lambda$ spectrum) is zero at zero frequency or infinite wavelength, whereas the variable P (for the λ spectrum) is zero for zero wavelength or infinite frequency. Consequently, dP' is equal to –dP. The prime on dP' is henceforth dropped. Applying Eq. 11 to Eq. 10 gives the power spectrum in terms of $(1/\lambda)$:

$$dP/d(1/\lambda) = [56.39\lambda_h^2 TP]/\{(\lambda/\lambda_h)^3(\exp[3.501(\lambda_h/\lambda)] - 1)\} \;(\text{cm-°K})^{-} \qquad (12)$$

By Eq. 7, $\lambda_h T$ is equal to 0.4107 cm-°K, and so this can be expressed as

$$\begin{aligned} dP/d(1/\lambda) &= [23.16\,\lambda_h\, P]/\{(\lambda/\lambda_h)^3(\exp[3.501(\lambda_h/\lambda)] - 1)\} \\ &= [\,9.512(P/T)\text{ cm-°K}]/\{(\lambda/\lambda_h)^3(\exp[3.501(\lambda_h/\lambda)] - 1)\} \qquad (13) \end{aligned}$$

This can also be expressed as follows in terms of the frequency ratio f/f_h, where f_h is the frequency at the wavelength λ_h:

$$dP/d(1/\lambda) = [9.512\,(P/T)(f/f_h)^3]/\{\exp[3.501(f/f_h)] - 1)\} \;\text{cm-°K} \qquad (14)$$

Setting the derivative of this equal to zero shows that its maximum (peak) value occurs at the frequency

$$f_p = 0.8059\, f_h \qquad (15)$$

The maximum value of Eq. 14 (at this frequency f_p) is

$$\text{Max}[dP/d(1/\lambda)] = 0.3151\,(P/T)\text{ cm-°K} \qquad (16)$$

Comparing Eqs. 7, 15 shows that this maximum ("peak") value occurs at a wavelength λ_p given by

$$\lambda_p T = 0.5096 \text{ cm-°K} \qquad (17)$$

Equation 14 can be approximated quite accurately, except at low frequencies, by ignoring the -1 term in the denominator. At the frequency f_h , the exact denominator is 32.15, and the denominator in the approximation is 33.15. The numerator is multiplied by the ratio (33.15/32.15) to obtain the following approximation, which matches the original equation exactly at the frequency f_h:

$$dP/d(1/\lambda) = [9.81 (P/T)(f/f_h)^3]/\{\exp[3.501(f/f_h)]\} \text{ cm-°K} \qquad (18)$$

3, Spectrum in Terms of Photon Rate

Let us denote the photon rate in photons per second as N*. The photon rate over a small frequency band is denoted ΔN^*. The energy of a photon is denoted u_p and is equal to

$$u_p = hf = hc/\lambda \qquad (19)$$

where Planck's constant h is given in Ref [8], page 7-3, as

$$h = 6.6251\text{x}10^{-27} \text{ erg-sec} \qquad (20)$$

The photon rate within a small interval $\Delta(1/\lambda)$ is equal to

$$\Delta N^* = [\Delta P/u_p] = (1/u_p)[dP/d(1/\lambda)]\Delta(1/\lambda) \qquad (21)$$

Energy u_p is equal to $u_{p(h)}(f/f_h)$, where $u_{p(h)}$ is the photon energy at the frequency f_h; and the increment $\Delta(1/\lambda)$ is equal to $(1/\lambda_h)\Delta(f/f_h)$. Hence Eq. 21 can be expressed as

$$u_{p(h)}\Delta N^* = (f_h/f)[dP/d(1/\lambda)](1/\lambda_h)\,\Delta(f/f_h) \qquad (22)$$

Substitute into Eq. 22 the approximation given in Eq. 18 to obtain

$$u_{p(h)}\Delta N^* \approx [9.81(P/\lambda_h T)(f/f_h)^2\Delta(f/f_h)]/\{\exp[3.501(f/f_h)]\} \text{ cm-°K} \qquad (23)$$

Integrate this to obtain the total photon rate N*

$$u_{p(h)}\Delta N^* \approx [9.81(P/\lambda_h T)\text{cm-°K}] \int \{(f/f_h)^2]/\exp[3.501(f/f_h)]\}d(f/f_h) \qquad (24)$$

The integral is performed over limits from zero to infinity. In accordance with Eq. 7, replace $\lambda_h T$ by (0.4107 cm-°K). Define the exponent $3.501(f/f_h)$ as the variable x. In terms of the variable x, Eq. 24 becomes

$$u_{p(h)} N^* \approx 0.5566\ P \int x^2 e^{-x} dx \qquad (25)$$

It can be shown that the integral (from zero to infinity) is equal to 2, and so Eq. 24 becomes

$$N^* \approx 2(0.5566)\ P/u_{p(h)} = 1.113\ P/u_{p(h)} \qquad (26)$$

Define $u_{p(h)}/1.113$ as the parameter u_{bb} . Equation 26 becomes

$$N^* = P/u_{bb} \qquad (27)$$

This appendix treats this as an exact relation, but the analysis using the blackbody spectrum from which our value of u_{bb} was derived was not exact. Nevertheless the error in the analysis should not be important. The wavelength λ_{bb} is defined in terms of u_{bb} by

$$u_{bb} = hc/\lambda_{bb} \qquad (28)$$

Apply the value for $\lambda_h T$ in Eq. 7, noting that $u_{p(h)}$ is equal to 1.113 u_{bb}. The corresponding value for λ_{bb} is

$$\lambda_{bb} T = 0.457 \text{ cm-°K} \qquad (29)$$

Thus, one can compute from Eq. 29 the effective photon wavelength λ_{bb} for a blackbody radiator of temperature T, and from Eq. 28 one can compute the effective photon energy u_{bb} for that blackbody radiator. Equation A7 shows that the photon rate from the blackbody is equal to the power P radiated from the blackbody divided by u_{bb}.

4, Intensity of COBE Cosmic Microwave Radiation

In 1989, the Cosmic Background Explorer (COBE) satellite made accurate measurements of the cosmic background radiation at wavelengths from 0.05 cm to 1 cm. Strong signals were detected, coming almost uniformly from all directions, with a spectrum that accurately matches a blackbody at a temperature of 2.73 °K. Narlikar [6] shows on page 328 a plot of this spectrum versus reciprocal wavelength $(1/\lambda)$. The following maximum value was read from the plot:

$$\text{Max}\{dP/d(1/\lambda)\} = 1.16\text{x}10^{-4} A\Omega\ [(\text{erg/sec})/\text{cm}^2]/(\text{steradian-cm}^{-1}) \qquad (30)$$

where P is power (erg/second), A is the capture area of the receiver (cm^2), and Ω is the acceptance solid angle of the receiver (steradians). Equation 16 showed that the maximum value of a blackbody power spectrum versus (1/λ) is equal to 0.3151 P/T cm-°K, which is (0.1154 P cm) for a blackbody temperature T of 2.735 °K. Setting Eq. 30 equal to (0.1154 P cm) gives the total power P of the COBE spectrum:

$$P = 1.005\text{x}10^{-3}\ A\Omega\ [(\text{erg/sec})/\text{cm}^2]/\text{steradian} \quad (31)$$

This is the power received within a small acceptance solid angle Ω. When this solid angle Ω is set equal to π steradians, the equation gives the total power falling on a surface in space of area A. The power per surface area is

$$P/A = 1.005\text{x}10^{-3}\ \pi\ [(\text{erg/sec})/\text{cm}^2] = 3.157\text{x}10^{-10}\ \text{watt/cm}^2 \quad (32)$$

This is expressed in watts, where 1 watt = 10^7 erg/sec. Equation 3 gave a general expression for the power per unit area radiated from the surface of an ideal blackbody of temperature T. Setting this temperature T equal to 2.73 ° K gives

$$P/A = 3.154\text{x}10^{10}\ \text{watt/cm}^2 \quad \textit{(Ideal blackbody at 2.735 ° K):} \quad (33)$$

The values of Eqs. 32, 33 are practically identical. Therefore, the blackbody microwave radiation level measured by the COBE satellite is the same as the level that would be radiated from the surface of an ideal blackbody of the measured blackbody temperature 2.73 ° K.

5, Photon Rate for Solar Radiation

Let us consider the radiation from the sun. In *Believe* [1], Table 11-3 of Chapter 11 gives the following for the luminosity of the sun, which is the power radiated from the sun:

$$L_{sun} = P_{sun} = 3.826\text{x}10^{33}\ \text{erg/sec} = 3.826\text{x}10^{26}\ \text{watt} \quad (34)$$

This was denoted L_{sun} in *Believe* [1], but here is denoted P_{sun}. The diameter of the sun, which is denoted D_{sun}, is 1,392,530 kilometers (km). Hence, the surface area of the sun is

$$A_{sun} = \pi\, D_{sun}{}^2 = 6.092\text{x}10^{12}\ \text{km}^2 = 6.092\text{x}10^{18}\ \text{meter}^2 \quad (35)$$

Dividing Eq. 34 by Eq. 35 gives the power per unit surface area radiated from the sun:

$$P_{sun}/A_{sun} = 62.8\text{x}10^{6}\ \text{watt/meter}^2 = 6280\ \text{watt/cm}^2 \quad (36)$$

The sun radiation can be approximated as an ideal blackbody. Equation 3 gave the following for power per unit-area radiated from a blackbody of temperature T

$$P/A = 5.679\text{x}10^{-12}\ T^4\ (\text{watt/cm}^2)/°\text{K}^4 \quad (37)$$

Setting Eq. 37 equal to 36 gives the following for the equivalent blackbody temperature of solar radiation, which we denote T_{sun}

$$T_{sun} = [6280/(5.679x10^{-12})]^{1/4} = 5770\ °K \qquad (38)$$

Reference [7], page 43, states that the surface temperature of the sun is 10,430 ° F, which is 5777 ° C or 6050 ° K. Since this is only 5 percent greater than the ideal blackbody temperature (5770 ° K), the solar radiation closely approximates the radiation from an ideal blackbody.

Equations 27 to 29 showed how to calculate the photon rate of the radiation from a blackbody when the total power P and blackbody temperature T are known. Applying these principles gives the following for the effective photon wavelength λ_{bb} of solar blackbody radiation

$$(\lambda_{bb})_{sun} = (0.458\ cm\text{-}°K)/T = 7.94x10^{-5}\ cm \qquad (39)$$

The blackbody temperature T of the sun was set equal to 5770 °K. The energy of a photon at this wavelength, which is denoted u_{bb}, is computed from

$$(u_{bb})_{sun} = hc/(\lambda_{bb})_{sun} = 2.50x10^{-12}\ erg \qquad (40)$$

where h is Planck's constant (6.6251x10^{-27} erg-sec) and c is the speed of light (3x10^{10} cm/sec). As was shown in Eq. 27, the photon rate N* of the power radiated from a blackbody is equal to P/u_{bb} . Divide the solar radiated power P_{sun} of Eq. 34 by the solar blackbody photon energy $(u_{bb})_{sun}$ of Eq. 40 gives the following photon rate N* for radiation from the sun

$$N^*_{sun} = P_{sun}/(u_{bb})_{sun} = 1.530x10^{45}\ photon/sec \qquad (41)$$

6, Photon Density for Blackbody Radiator

Equation 3 gave the following for the power per unit area radiated by a blackbody:

$$P/A = 5.679x10^{-5}\,T^4\ (erg/sec)/cm^2\text{-}°K^4 \qquad (42)$$

This has been converted from watts to erg/sec by noting that 1 watt is equal to 10^{7} erg/sec. From Eqs. 27, 28, the photon rate per unit area is equal to

$$N^*/A = (P/A)/u_{bb} = [(P/A)\,\lambda_{bb}]/(hc) \qquad (43)$$

Substitute into this the expression for $\lambda_{bb}T$ in Eq. 29:

$$N^*/A = [(P/A)\,\lambda_{bb}]/(hc) = [(P/A)(0.457\ cm\text{-}°K)]/(hcT) \qquad (44)$$

Combine Eqs. 42, 44

$$N^*/A = [2.595\text{x}10^{-5}\ T^3]/(hc) \qquad (\text{erg/sec})/(\text{cm-°K}^3) \qquad (45)$$

Substitute into this the speed of light c ($3\text{x}10^{10}$ cm/sec) and the value of Planck's constant h given in Eq. 20:

$$N^*/A = 1.3056\text{x}10^{11}\ T^3 \ \ (\text{photon/sec})/(\text{cm}^2\text{- ° K}^3) \qquad (46)$$

This is a general expression for the intensity of the radiation from an ideal blackbody, expressed in terms of photon rate.

An ideal blackbody radiator operates under conditions of thermal equilibrium, and under this condition the photons are packed as closely as is physically possible. Let us define the parameter λ_{eff} such that the radiation from an ideal blackbody has one photon within a cube of dimensions λ_{eff} on a side. The number of photons within a cube of thickness λ_{eff} is equal to

$$N = (N^*/A)(\lambda_{eff})^2\ (\lambda_{eff}/c) = (N^*/A)(\lambda_{eff})^3/c \qquad (47)$$

Since $(\lambda_{eff})^2$ is the cross section area of the cube, the expression $(N^*/A)(\lambda_{eff})^2$ is the rate at which photons enter the cube. This rate is multiplied by (λ_{eff}/c), which is the time for light to travel the thickness of the cube, to obtain the number of photons N within the cube. Set this number N equal to unity, and solve for λ_{eff}:

$$(\lambda_{eff})^3 = c/(N^*/A) \qquad (48)$$

Substitute into this the expression for N*/A in Eq. 46, and the value for the speed of light c.

$$(\lambda_{eff})^3 = 0.2297/T^3 \ \ (\text{cm- ° K})^3 \qquad (49)$$

Taking the cube root of this gives

$$\lambda_{eff}T = 0.612 \ \ \text{cm- ° K} \qquad (50)$$

Equation 7 showed that $\lambda_h T$ is equal to 0.4017 cm-°K. Hence λ_{eff} is related as follows to λ_h:

$$\lambda_{eff} = 1.49\ \lambda_h \qquad (51)$$

Remember that λ_h is the half power wavelength of the blackbody spectrum: half of the power lies below λ_h and half lies above it. This result can be summarized by

A blackbody radiates the maximum possible photon rate for thermal equilibrium at its wavelength band. This photon rate is equivalent to one photon within a cube having a thickness of λ_{eff} , which is *approximately* equal to *1.50* λ_h *(where* λ_h *is the half-power wavelength of the spectrum}. About ¾ of the power lies below the wavelength* λ_{eff} .

7, Blackbody Spectrum Expressed in Terms of Frequency

This section derives a formula for the blackbody spectrum expressed in terms of frequency, which is used to calculate Cosmic Background Radiation. Equation 14 gives the following for the spectrum of an ideal blackbody expressed in terms of reciprocal wavelength:

$$dP/d(1/\lambda) = \{K_{bb1}(P/T)(f/f_h)^3\}/\{\exp[3.501(f/f_h)] - 1\} \qquad (52)$$

$$K_{bb1} = 9.512 \text{ cm-°K} \qquad (53)$$

Equation 7 gives

$$\lambda_h T = 0.4107 \text{ cm-°K} \qquad (54)$$

Since $\lambda_h/\lambda = f/f_h$, we can set the following in Eq 51:

$$dP/d(1/\lambda) = [\lambda_h\, dP/d(1/\lambda)] = [\lambda_h\, dP/d(f/f_h)] \qquad (55)$$

Multiply Eq. 52 by (T/P) and substitute Eq 55 into this:

$$(T/P)dP/d(1/\lambda) = \{\{\lambda_h T)(dP/P)\}/d(f/f_h) = \{K_{bb1}(f/f_h)^3\}/\{\exp[3.501(f/f_h)] - 1\} \qquad (56)$$

This can be expressed as

$$(dP/P)\}/d(f/f_h) = \{K_{bb1}/(\lambda_h T)\}(f/f_h)^3/\{\exp[3.501(f/f_h)] - 1\} \qquad (57)$$

The constant K_{bb} is defined by

$$K_{bb} = \{(\lambda_h T)/K_{bb1}\}[\exp(3.501) - 1] \qquad (58)$$

Applying Eqs 53 and 54 gives

$$K_{bb} = (0.4107/9.512)[\exp(3.501) - 1] = 1.388 \qquad (59)$$

Applying this to Eq. 57 gives

$$(dP/df) = \{P/K_{bb}f_h\}(f/f_h)^3\{[\exp(3.501) - 1]/[\exp[3.501(f/f_h)] - 1\}\} \qquad (60)$$

At the frequency f_h , dP/df becomes

$$[dP/df] \text{ at } f_h = P/K_{bb}f_h \quad (61)$$

We call the frequency f_h the *half-power frequency*, because half of the spectral power lies at lower frequencies, and half lies at higher frequencies. Probably a better term would be *mid-power frequency*. The corresponding wavelength parameter is denoted λ_h , which is the *half-power* (or *mid-power*) *wavelength*. Half of the power lies at wavelengths less than λ_h , and half lies at greater wavelengths. Equation 60 can be expressed as

$$(dP/df) = \{[dP/df]_h\}K_0(f/f_h)^3/\{[\exp[3.501(f/f_h)] - 1\} \quad (62)$$

Where $[dP/df]_h$ represents the value of (dP/df) at the frequency f_h given by Eq 61, and K_0 denotes the following constant

$$K_0 = [\exp(3.501) - 1] = 32.149 \quad (63)$$

References

[1] Adrian Bjornson, *A Universe that We Can Believe, Addison :Press, 2000, described in world-wide website Olduniverse.com* ISBN 09703231-0-7.

[5] Dwight C. Gray, *American Institute of Physics Handbook*, Second Edition, McGraw-Hill, New York, 1963

[6] J. V. Narlikar, *Introduction to Cosmology*, Cambridge University Press, Cambridge, England, Second Edition, 1993, ISBN 0-521-41250, ISBN 0-521-42352 (paperback).

[7] Kevin Krisciunas and Bill Yenne, *The Pictorial Atlas of the Universe*. Mallard Press, 1989, ISBN 0-792-45200-3.

[8] Paul Marmet, "A New Non-Doppler Redshift", in website www.newtonphysics.on.ca, specific website address is www.newtonphysics.on.ca/HUBBLE/Hubble.html.