5,C Addendum Appendix C
Calculation of Energy-Momentum Tensor

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This appendix describes the general principles for calculating the energy-momentum tensor. It gives the detailed analytical steps that Schwartzschild developed to form the energy-momentum tensor that he used in his solution of the Einstein theory.

1, Definition of Energy Momentum Tensor $T^{\mu\nu}$

Tolman [5] defines the energy-momentum tensor by considering its elements in ***proper coordinates***, which are coordinates that move with the material at each point of interest. Within a small region of proper coordinates, all velocities are zero. Tolman [5] (p. 215, Eq. 85.1) gives the following matrix for the energy-momentum tensor in proper coordinates:

$$T_p^{\mu\nu} = \begin{vmatrix} \rho & 0 & 0 & 0 \\ 0 & p_{xx} & p_{xy} & p_{xz} \\ 0 & p_{yx} & p_{yy} & p_{yz} \\ 0 & p_{zx} & p_{zy} & p_{zz} \end{vmatrix} \quad (1)$$

In this matrix the variable ρ is the *mass density*, which is mass per unit volume. The variable p_{xy} is the *stress*, which is force per unit area. As shown in Appendix A of *1,2 Simple Explanation of Einstein's Relativity Theory*, p_{xy} is force per unit area applied in the x-direction to a surface that is perpendicular to the y-direction. The mass density ρ and the pressure p are expressed in normalized mass units. The conventional mass M is multiplied by G/c^2 to obtain the normalized mass m. Therefore the elements in the above matrix should be multiplied by G/c^2 if ρ and p are expressed in conventional mass units.

2, Formula for Converting Coordinates of a Tensor

The matrix of Eq. 1 is a general expression that can be applied at any point in any physical medium. However each point of a general medium may be moving at a different velocity, and so each point may require its own set of proper coordinates. In order to utilize the information for the whole body, one must convert the tensor values for each point to a single coordinate system. This is achieved by applying the general formula for translating a tensor from one coordinate system to another

The following formula allows one to translate a contravariant tensor from proper coordinates to any other coordinate system:

$$T^{\mu\nu} = \Sigma_\alpha \Sigma_\beta \left(\partial x^\mu / \partial x_p^{\alpha}\right)\left(\partial x^\nu / \partial x_p^{\beta}\right) T_p^{\alpha\beta} \quad (2)$$

Variables with the subscript p apply to proper coordinates. Variables without subscripts apply to the fixed coordinate system into which the data are being translated. (Equivalent formulas apply to covariant and mixed tensors.)

This formula of Eq. 2 is the key for implementing the Ricci-Riemann calculus of curved space. All ***true tensors*** can be transformed from one coordinate system to another in accordance with this formula (or its equivalent). The subscript p can apply to any coordinate system in which a tensor $T_p^{\alpha\beta}$ is specified, and the variables without subscripts can apply to any other coordinate system. ***By means of this formula, Einstein achieved covariance in his physical law of general relativity by specifying his law in terms of true tensors.***

Since Eq. 2 has two summations, performed over the four values of the α and β indices, the equation for the $T^{\mu\nu}$ tensor yields 16 terms for each element of the tensor. Hence 160 terms are required to specify the ten independent elements of the $T^{\mu\nu}$ tensor. To illustrate the application of the tensor formula of Eq. 2, the following are 8 terms of the 16-term expansion, for $\alpha = 0, 1$, and for $\beta = 0, 1, 2, 3$:

$$\begin{aligned} T^{\mu\nu} = {} & (\partial x^\mu/\partial x_p{}^0)(\partial x^\nu/\partial x_p{}^0)T_p{}^{00} + (\partial x^\mu/\partial x_p{}^0)(\partial x^\nu/\partial x_p{}^1)T_p{}^{01} \\ & + (\partial x^\mu/\partial x_p{}^0)(\partial x^\nu/\partial x_p{}^2)T_p{}^{02} + (\partial x^\mu/\partial x_p{}^0)(\partial x^\nu/\partial x_p{}^3)T_p{}^{03} \\ & + (\partial x^\mu/\partial x_p{}^1)(\partial x^\nu/\partial x_p{}^0)T_p{}^{10} + (\partial x^\mu/\partial x_p{}^1)(\partial x^\nu/\partial x_p{}^1)T_p{}^{11} \\ & + (\partial x^\mu/\partial x_p{}^1)(\partial x^\nu/\partial x_p{}^2)T_p{}^{12} + (\partial x^\mu/\partial x_p{}^1)(\partial x^\nu/\partial x_p{}^3)T_p{}^{13} + \text{etc.} \end{aligned} \qquad (3)$$

Both of the indices μ, ν are set equal to 0, 1, 2, 3 to obtain the specific formulas for the individual elements of the energy-momentum tensor $T^{\mu\nu}$. A total of 256 terms are needed to calculate all 16 elements of this tensor. Since this tensor is symmetric, only 10 elements need be computed. This reduces the number of terms to 160.

The expressions of the form $\partial x^\nu/\partial x_p{}^2$ in Eq. 3 are calculated by examining the equations that specify the two sets of coordinates (proper and fixed). By applying calculus to those equations, partial derivatives are computed that relate the variables that are specified in the two sets of coordinates.

An extremely tedious calculation is required to compute the 160 terms that are needed to translate an energy-momentum tensor from proper to fixed coordinates. It may be possible to implement this with hand computations for one point of the medium. However for a general medium, where the velocity changes from point to point, many different sets of proper coordinates must be used to characterize the medium with reasonable accuracy. A different set of transformation equations is required for each set of proper coordinates.

For general applications, computer calculations are essential to specify the energy-momentum tensor. Since computers were not available to Einstein, he and other scientists of his day had to restrict themselves to very simple physical models when calculating the energy-momentum tensor.

The Yilmaz theory of gravitation does not require that the energy-momentum tensor be calculated when the theory is applied. Therefore the practical problems of computing the

energy-momentum tensor are irrelevant when the Yilmaz theory is used, even for the general time-varying Yilmaz theory.

3, Calculating the Schwartzschild Energy-Momentum Tensor

In Eq. 1 the three stresses on the diagonal (p_{xx}, p_{yy}, p_{zz}) are the compressive stresses, which represent pressure in a fluid. The six nondiagonal stresses (p_{xy}, p_{xy}, p_{xy}, etc.) are the shear stresses. Schwartzschild modeled the star as a perfect fluid, which has no shear (or viscous) forces. The three pressure stresses (p_{xx}, p_{yy}, p_{zz}) in rectangular coordinates are equal, and each is represented by the hydrostatic pressure p. Hence, The Schwartzschild energy-momentum tensor in proper rectangular coordinates simplifies from Eq. 1 to

$$T_p^{\mu\nu} = \begin{vmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{vmatrix} \qquad (4)$$

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Converting from proper to fixed coordinates. Since $T_p^{\mu\nu}$ is diagonal, it has nonzero values only when $\nu = \mu$. Hence in the general formula of Eq. 2 for converting the coordinates of a tensor, we can set $\beta = \alpha$ in the expression for $T_p^{\alpha\beta}$. This eliminates the summation over the index β and Eq. 2 simplifies to

$$T^{\mu\nu} = \Sigma_\alpha \, (\partial x^\mu / \partial x_p^{\alpha})(\partial x^\nu / \partial x_p^{\alpha}) T_p^{\alpha\alpha} \qquad (5)$$

The summation over the index α can be separated into two parts: the first part is the value for $\alpha = 0$ and the second is the summation over $\alpha = 1, 2, 3$.

$$T^{\mu\nu} = (\partial x^\mu/\partial x_p^{0})(\partial x^\nu/\partial x_p^{0})T_p^{00} + \Sigma_k(\partial x^\mu/\partial x_p^{k})(\partial x^\nu/\partial x_p^{k})T_p^{kk} \qquad (6)$$

The index in the summation is changed from α to k, where k takes the three values (1, 2, 3). The matrix of Eq. 4 shows that the elements of the Schwartzschild energy-momentum tensor in proper coordinates are

$$T_p^{00} = \rho \quad ; \quad T_p^{11} = T_p^{22} = T_p^{33} = p \qquad (7)$$

Apply these values to Eq. 6, noting that T_p^{kk} is equal to p for all three values of k. This gives

$$T^{\mu\nu} = (\partial x^\mu/\partial x_p^{0})(\partial x^\nu/\partial x_p^{0})\rho + \Sigma_k \, (\partial x^\mu/\partial x_p^{k})(\partial x^\nu/\partial x_p^{k})p \qquad (8)$$

In proper coordinates, Special Relativity applies within a small region, and so the line element in proper coordinates is

$$ds^2 = d\tau^2 - dx^2 - dy^2 - dz^2$$

$$= g_{00}(dx^0)^2 + g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + g_{33}(dx^3)^2 \qquad (9)$$

Hence the metric tensor values for proper coordinates, which are the same as for special relativity, are

$$g_{00} = 1 \; ; \quad g_{11} = g_{22} = g_{33} = -1 \tag{10}$$

When the covariant metric tensor $g_{\mu\nu}$ is diagonal, the contravariant tensor $g^{\mu\nu}$ is also diagonal, and its diagonal elements are the reciprocals of the covariant elements. Hence the contravariant metric tensor values in proper coordinates are

$$g_p^{00} = 1 \; ; \quad g_p^{11} = g_p^{22} = g_p^{33} = -1 \tag{11}$$

The subscript p has been included here to emphasize the fact that these are the elements of the contravariant metric tensor in proper coordinates. Since $g_p^{\mu\nu}$ is diagonal, we can use a formula equivalent to Eq. 6 to translate the elements of $g_p^{\mu\nu}$ from proper coordinates to the fixed coordinates of the $T^{\mu\nu}$ tensor. This gives

$$g^{\mu\nu} = (\partial x^\mu/\partial x_p^0)(\partial x^\nu/\partial x_p^0)g_p^{00} + \Sigma_k\,(\partial x^\mu/\partial x_p^k)(\partial x^\nu/\partial x_p^k)g_p^{kk} \tag{12}$$

Apply to this the values for $g_p^{\mu\nu}$ in Eq. 11

$$g^{\mu\nu} = (\partial x^\mu/\partial x_p^0)(\partial x^\nu/\partial x_p^0) - \Sigma_k\,(\partial x^\mu/\partial x_p^k)(\partial x^\nu/\partial x_p^k) \tag{13}$$

Multiply each term by pressure p to obtain

$$g^{\mu\nu}p = (\partial x^\mu/\partial x_p^0)(\partial x^\nu/\partial x_p^0)p - \Sigma_k\,(\partial x^\mu/\partial x_p^k)(\partial x^\nu/\partial x_p^k)p \tag{14}$$

Add this to $T^{\mu\nu}$ in Eq. 8. The Σ_k summation terms cancel to give

$$T^{\mu\nu} + g^{\mu\nu}\,p = (\partial x^\mu/\partial x_p^0)(\partial x^\nu/\partial x_p^0)\,(\rho + p) \tag{15}$$

Tolman [5] explains (in p. 217, Eq. 85.6) that the partial derivative $\partial x^\mu/\partial x_p^0$ is equal to the ordinary derivative dx^μ/ds taken along the ds line-element path. Hence Eq. 15 can be expressed as

$$T^{\mu\nu} + g^{\mu\nu}\,p = (dx^\mu/ds)(dx^\nu/ds)\,(\rho + p) \tag{16}$$

Solving for $T^{\mu\nu}$ gives

$$T^{\mu\nu} = (dx^\mu/ds)(dx^\nu/ds)\,(\rho + p) - g^{\mu\nu}\,p \tag{17}$$

Tolman [5] gives this formula on page 217 (Eq. 87.5) and on page 243 (Eq. 97.5).

Mixed form of energy-momentum tensor. We want to convert this formula for the energy-momentum tensor $T^{\mu\nu}$ to the mixed form $T_\mu{}^\nu$. As shown in Appendix A of *Believe* [1], for a diagonal metric tensor this is calculated from

$$T_\mu{}^\nu \ = \ g_{\mu\mu}\, T^{\mu\nu} \qquad (18)$$

Applying this to Eq. 17 gives

$$T_\mu{}^\nu \ = \ (dx^\mu/ds)(dx^\nu/ds)\, g_{\mu\mu}\, (\rho + p) - g_\mu{}^\nu\, p \qquad (19)$$

It can be shown that the mixed metric tensor $g_\mu{}^\nu$ is equal to the Kronecker delta $\delta_\mu{}^\nu$, which is unity when the indices are equal and is zero when they are unequal.

` To calculate the elements of $T_\mu{}^\nu$ from Eq. 19, we must determine the dx^μ/ds derivatives. As shown by Tolman [5] (p. 243, Eq. 95.9), the Schwartzschild solution assumes that the medium within the star is not moving and so the derivatives of R, θ, ψ are zero:

$$dR/ds \ = \ d\theta/ds \ = \ d\psi/ds \ = \ 0 \qquad (20)$$

Since the Schwartzschild metric tensor is diagonal, the general form of the Schwartzschild line element in polar coordinates is

$$ds^2 \ = \ g_{00}\, d\tau^2 + g_{11}\, dR^2 + g_{22}\, d\theta^2 + g_{33}\, d\psi^2 \qquad (21)$$

By Eq. 20, set dR, dθ, dψ equal to zero and solve Eq. 21 for dτ/ds. This gives

$$d\tau/ds. \ = \ 1/\sqrt{[g_{00}]} \qquad (22)$$

Since $\tau = x^0$, this can be expressed as

$$dx^0/ds \ = \ 1/\sqrt{[g_{00}]} \qquad (23)$$

Since R, θ, ψ represent x^1, x^2, x^3, Eq. 20 can be expressed as

$$dx^1/ds \ = \ dx^2/ds \ = \ dx^3/ds \ = \ 0 \qquad (24)$$

Applying Eq. 24 to Eq. 19 shows that (if either μ or ν is 1, 2, or 3), the first term of Eq. 19 is zero and the equation becomes

$$T_\mu{}^\nu \ = \ - \delta_\mu{}^\nu\, p \qquad \text{if } \mu \text{ or } \nu = 1, 2, 3 \qquad (25)$$

Remember that $g_\mu{}^\nu$ is equal to $\delta_\mu{}^\nu$. Since $\delta_\mu{}^\nu$ in zero if $(\mu \neq \nu)$, and is unity if $(\mu = \nu)$, Eq. 25 can be expressed as

$$T_\mu{}^\nu \ = \ 0 \qquad \text{if } \mu \neq \nu \qquad (26)$$

$$T_\mu{}^\mu = - p \qquad \text{if } \nu = \mu = 1, 2, 3 \tag{27}$$

If neither μ or ν is 1, 2, or 3, then both μ and ν must be zero. Setting $\mu = \nu = 0$ in Eq. 19 gives

$$T_0{}^0 = (dx^0/ds)(dx^0/ds)\, g_{00}\, (\rho + p) - \delta_0{}^0\, p \tag{28}$$

Set $\delta_0{}^0$ equal to unity, and by Eq. 23 set dx^0/ds equal to $1/\sqrt{[g_{00}]}$. This gives

$$T_0{}^0 = (\rho + p) - p = \rho \tag{29}$$

Final Schwartzschild energy-momentum tensor. Equation 26 shows that the energy-momentum tensor for the Schwartzschild solution is diagonal. Equations 27 and 29 show that it has the following diagonal elements:

$$T_0{}^0 = \rho \qquad ; \quad T_1{}^1 = T_2{}^2 = T_3{}^3 = - p \tag{30}$$

This discussion has described the detailed steps for computing the elements of the energy-momentum tensor for the Schwartzschild solution.

References

[1] Adrian Bjornson, *A Universe that We Can Believe, Addison :Press, 2000, described in world-wide website Olduniverse.com* ISBN 09703231-0-7.

[5] Richard C. Tolman, *Relativity, Thermodynamics, and Cosmology*, Dover Publications (31 East 2nd St., Mineola, NY, 11501), 1987 (first published, 1934, Oxford, Clarendon Press) ISBN 0-486-65383-8.