5,D Addendum Appendix D
Analysis of Geodesic Equations

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This appendix supplements Chapter 4, which discusses the application of the geodesic equations to the Yilmaz theory. The following summarizes the material in this appendix.

Section 1 gives the general formula for the geodesic equations. It derives from this the general formulas for the second derivatives in spherical coordinates that correspond to a static, spherically symmetric gravitational field. Section 2 applies the results of Section 1 to the Yilmaz theory. It calculates the four specific formulas for the second derivatives in polar coordinates for the Yilmaz solution of a single star and the Yilmaz cosmology model. Section 3 shows that the geodesic equation for time can be solved directly. Hence, the geodesic equations relative to time are expressed in terms of the first derivative of time.

Section 4 derives formulas for the geodesic equations of the general static Yilmaz theory expressed in rectangular coordinates. Since these equations do not require a gravitational field with spherical symmetry, they can be applied to multi-body applications. For example, they can be used to calculate the accurate orbits of planets and other bodies in our solar system.

Within our solar system, the derivative $d\tau/ds$ along a geodesic trajectory can be very accurately approximated by considering only the gravitational field produced by the sun. The reason for this is that $d\tau/ds$ is very close to unity. By applying this approximation to the geodesic equations derived in Section 4, much simpler geodesic equations are derived in Section 5 that can be used to calculate the orbits of bodies in our solar system.

1, Geodesic Equations for a Static, Spherically Symmetric Gravitational Field

This section considers the general formula that specifies the geodesic equations. It derives from this the four geodesic equations in spherical coordinates that apply to a static, spherically symmetric gravitational field. These four equations specify the second derivatives of the four space-time variables in spherical coordinates (τ, R, θ, ψ), where these derivatives are with respect to differential motion ds along the path of the body.

The general formula for the geodesic equations is given as follows by Tolman [5] (p. 495, Eq. 20):

$$d^2x^\alpha/ds^2 \ = \ - \ \Sigma_\mu \, \Sigma_\nu \, \Gamma_{\mu\nu}{}^\alpha \, u^\mu \, u^\nu \qquad (1)$$

where the variable u^μ represents the following derivativ

$$u^\mu \ = \ dx^\mu/ds \qquad (2)$$

The two summations shown in Eq. 1 are not stated implicitly in Tolman's equation. However, they are implied, because the indices μ and ν are repeated. We will solve Eq. 1 for values of the index α equal to 0, 1, 2, 3.

Table 1: Non-zero Christoffel Symbols for Static, Spherically Symmetric Gravitational Fiel

$\Gamma_{01}{}^{0} = \Gamma_{10}{}^{0} \quad \Gamma_{22}{}^{1} \quad \Gamma_{33}{}^{2}$
$\Gamma_{00}{}^{1} \quad \Gamma_{33}{}^{1} \quad \Gamma_{31}{}^{3} = \Gamma_{13}{}^{3}$
$\Gamma_{11}{}^{1} \quad \Gamma_{21}{}^{2} = \Gamma_{12}{}^{2} \quad \Gamma_{32}{}^{3} = \Gamma_{23}{}^{3}$

Table 1 lists the non-zero Christoffel symbols when the gravitational field is static and spherically symmetric. The following discussion applies this information to Eq. 1, and derives the four geodesic equations for this case. When we consider only the Christoffel symbols given in Table 1, the Geodesic formula of Eq. 1 yields the following equations when the index a is set equal to 0, 1, 2, 3:

$$d^2x^0/ds^2 = - \{ \Gamma_{10}{}^{0} u^1 u^0 + \Gamma_{01}{}^{0} u^0 u^1 \} \tag{3}$$

$$d^2x^1/ds^2 = - \{\Gamma_{00}{}^{1}u^0u^0 + \Gamma_{11}{}^{1}u^1u^1 + \Gamma_{22}{}^{1}u^2u^2 + \Gamma_{33}{}^{1}u^3u^3\} \tag{4}$$

$$d^2x^2/ds^2 = - \{ \Gamma_{12}{}^{2} u^1u^2 + \Gamma_{21}{}^{2} u^2 u^1 + \Gamma_{33}{}^{2} u^3 u^3 \} \tag{5}$$

$$d^2x^3/ds^2 = - (\Gamma_{13}{}^{3}u^1u^3 + \Gamma_{31}{}^{3}u^3u^1 + \Gamma_{23}{}^{3}u^2u^3 + \Gamma_{32}{}^{3}u^3u^2\} \tag{6}$$

In Eqs. 1 to 6, apply the equality relations for the Christoffel symbols given in Table 1 and combine terms. This yields

$$d^2x^0/ds^2 = - 2 \Gamma_{01}{}^{0} u^0u^1 \tag{7}$$

$$d^2x^1/ds^2 = - \Gamma_{00}{}^{1} (u^0)^2 - \Gamma_{11}{}^{1}(u^1)^2 - \Gamma_{22}{}^{1}(u^2)^2 - \Gamma_{33}{}^{1}(u^3)^2 \tag{8}$$

$$d^2x^2/ds^2 = - 2 \Gamma_{12}{}^{2} u^1u^2 - \Gamma_{33}{}^{2} (u^3)^2 \tag{9}$$

$$d^2x^3/ds^2 = - 2 \Gamma_{13}{}^{3} u^1 u^3 - 2 \Gamma_{23}{}^{3} u^2 u^3 \tag{10}$$

Replace the generic x^μ coordinates by the following specific names for the spherical coordinates:

$$x^0 = \tau, \; x^1 = r, \; x^2 = \theta, \; x^3 = \psi \tag{11}$$

In accordance with Eq. 2, replace the u^μ variables by the following

$$u^0 = dx^0/ds = d\tau/ds \tag{12}$$

$$u^1 = dx^1/ds = dr/ds \tag{13}$$

$$u^2 = dx^2/ds = d\theta/ds \qquad (14)$$

$$u^3 = dx^3/ds = d\psi/ds \qquad (15)$$

Equations 7 to 10 become

$$d^2\tau/ds^2 = -2\,\Gamma_{01}{}^0\,(d\tau/ds)\,(dr/ds) \qquad (16)$$

$$d^2r/ds^2 = -\Gamma_{00}{}^1\,(d\tau/ds)^2 - \Gamma_{11}{}^1(dr/ds)^2 - \Gamma_{22}{}^1(d\theta/ds)^2 - \Gamma_{33}{}^1(d\psi/ds)^2 \qquad (17)$$

$$d^2\theta/ds^2 = -2\,\Gamma_{12}{}^2\,(dr/ds)\,(d\theta/ds) - \Gamma_{33}{}^2\,(d\psi/ds)^2 \qquad (18)$$

$$d^2\psi/ds^2 = -2\,\Gamma_{13}{}^3\,(dr/ds)(d\psi/ds) - 2\,\Gamma_{23}{}^3\,(d\theta/ds)(d\psi/ds) \qquad (19)$$

These are the general expressions for the geodesic equations in spherical coordinates for a static spherically symmetric gravitational field.

2, Geodesic Equations for Spherically Symmetric Model of Yilmaz Theory

Let us apply these geodesic equations to the Yilmaz theory. Table 2 gives the formulas for the Christoffel symbols of the Yilmaz theory that apply to a static, spherically symmetric gravitational field, which were obtained from Table B-2 of Appendix B. Applying these values to Eqs. 16 to 19 gives the following geodesic equations:

$$d^2\tau/ds^2 = 2\,\partial_1\phi\,(d\tau/ds)\,(dr/ds) \qquad (20)$$

$$\begin{aligned} d^2r/ds^2 = {} & e^{-4\phi}\,\partial_1\phi\,(d\tau/ds)^2 - \partial_1\phi\,(dr/ds)^2 + r^2(\partial_1\phi + 1/r)(d\theta/ds)^2 \\ & + r^2\sin^2\theta(\partial_1\phi + 1/r)(d\psi/ds)^2 \qquad (21) \end{aligned}$$

$$d^2\theta/ds^2 = -2\,(\partial_1\phi + 1/r)(dr/ds)\,(d\theta/ds) + \sin\theta\,\cos\theta\,(d\psi/ds)^2 \qquad (22)$$

$$d^2\psi/ds^2 = -2\,(\partial_1\phi + 1/r]\,(dr/ds)\,(d\psi/ds) - 2\cot\theta\,(d\theta/ds)\,(d\psi/ds) \qquad (23)$$

Table-2: *General formulas for Christoffel Symbols of the Yilmaz theory, for a static, spherically symmetric gravitational field*

Symbol	Formula	Symbol	Formula
$\Gamma_{00}{}^1$	$-\,e^{-4\phi}\,\partial_1\phi$	$\Gamma_{01}{}^0 = \Gamma_{10}{}^0$	$-\,\partial_1\phi$
$\Gamma_{11}{}^1$	$\partial_1\phi$	$\Gamma_{21}{}^2 = \Gamma_{12}{}^2$	$(\partial_1\phi + 1/r)$
$\Gamma_{22}{}^1$	$-\,r^2(\partial_1\phi + 1/r)$	$\Gamma_{31}{}^3 = \Gamma_{13}{}^3$	$(\partial_1\phi + 1/r)$
$\Gamma_{33}{}^1$	$-\,r^2\sin^2\theta(\partial_1\phi + 1/r]$	$\Gamma_{32}{}^3 = \Gamma_{23}{}^3$	$\cot\theta$
$\Gamma_{33}{}^2$	$-\,\sin\theta\,\cos\theta$		

These geodesic equations can be simplified by replacing the differential angular variables by differential linear displacements in the θ and ψ directions, which are denoted dx_θ and dx_ψ and are equal to

$$dx_\theta \ = \ r\, d\theta \qquad (24)$$

$$dx_\psi \ = \ r \sin\theta\, d\psi \qquad (25)$$

Differentiating these gives the following second derivatives

$$d^2x_\theta \ = \ d(dx_\theta) \ = \ r\, d^2\theta \ + dr\, d\theta \qquad (26)$$

$$d^2x_\psi \ = \ d(dx_\psi) \ = \ r \sin\theta\, d^2\psi \ + \sin\theta\, dr\, d\psi + r \cos\theta\, d\theta\, d\psi \qquad (27)$$

Solving Eq. 26 for $(r\, d^2\theta)$ give

$$r\, d^2\theta \ = \ d^2x_\theta - dr\, d\theta \ = \ d^2x_\theta - (1/r)dr\, (rd\theta) \ = \ d^2x_\theta - (1/r)\, dr\, dx_\theta \qquad (28)$$

Solving Eq. 27 for $(r \sin\theta\, d^2\psi)$ gives

$$\begin{aligned} r \sin\theta\, d^2\psi \ &= \ d^2x_\psi - \sin\theta\, dr\, d\psi - r \cos\theta\, d\theta\, d\psi \\ &= \ d^2x_\psi - (1/r)\, dr\, (r \sin\theta\, d\psi) - (1/r) \cot\theta\, (r\, d\theta)(r \sin\theta\, d\psi) \\ &= \ d^2x_\psi - (1/r)\, dr\, dx_\psi - (1/r) \cot\theta\, dx_\theta\, dx_\psi \qquad (29) \end{aligned}$$

Let us use these relations to simplify the geodesic equations. Applying Eqs 24, 25 to Eq. 21 gives

$$\begin{aligned} d^2r/ds^2 \ &= \ e^{-4\phi}\, \partial_1\phi\, (d\tau/ds)^2 - \partial_1\phi\, (dr/ds)^2 + (\partial_1\phi + 1/r)\, \{ (rd\theta/ds)^2 + (r \sin\theta\, d\psi/ds)^2 \ \} \\ &= \ e^{-4\phi}\, \partial_1\phi(d\tau/ds)^2 - \partial_1\phi\, (dr/ds)^2 + (\partial_1\phi + 1/r)(dx_\theta/ds)^2 + (dx_\psi/ds)^2\} \qquad (30) \end{aligned}$$

Multiply Eq. 22 by r:

$$r\, d^2\theta/ds^2 \ = \ - 2\, (\partial_1\phi + 1/r)(dr/ds)\, (r\, d\theta/ds) \ + (1/r) \cot\theta\, (r \sin\theta\, d\psi/ds)^2 \qquad (31)$$

Applying Eqs. 24, 25, 28 to this gives

$$d^2x_\theta/ds^2 - (1/r)(dr/ds)(dx_\theta/ds) \ = \ - 2\, (\partial_1\phi + 1/r)(dr/ds)(dx_\theta/ds) + (1/r) \cot\theta\, (dx_\psi/ds)^2 \qquad (32)$$

Solving for d^2x_θ/ds^2 gives

$$d^2x_\theta/ds^2 \ = \ - (2\, \partial_1\phi + 1/r)(dr/ds)(dx_\theta/ds) + (1/r) \cot\theta\, (dx_\psi/ds)^2 \qquad (33)$$

Multiply Eq. 23 by $(r \sin\theta)$:

$$(r \sin\theta)(d^2\psi/ds^2) = - 2\,(\partial_1\phi + 1/r]\,(dr/ds)\,(r \sin\theta\, d\psi/ds) - (2/r) \cot\theta\,(r\, d\theta/ds)\,(r \sin\theta\, d\psi/ds) \qquad (34)$$

Applying Eqs. 24, 25, 29 to this gives

$$\begin{aligned} d^2x_\psi/ds^2 &- (1/r)\,(dr/ds)(dx_\psi/ds) - (1/r) \cot\theta\,(dx_\theta/ds)(dx_\psi/ds) \\ &= - 2\,(\partial_1\phi + 1/r]\,(dr/ds)\,(dx_\psi/ds) - (2/r) \cot\theta\,(dx_\theta/ds)\,(dx_\psi/ds) \qquad (35) \end{aligned}$$

Solving for d^2x_ψ/ds^2 gives

$$d^2x_\psi/ds^2 = - (2\partial_1\phi + 1/r]\,(dr/ds)\,(dx_\psi/ds) - (1/r) \cot\theta\,(dx_\theta/ds)\,(dx_\psi/ds) \qquad (36)$$

Further simplification of the geodesic equations is achieved by expressing the differential linear displacements in the θ and ψ directions (dx_θ, dx_ψ) by the total differential tangential displacement, which is denoted dx_t, and is given by

$$(dx_t)^2 = (dx_\theta)^2 + (dx_\theta)^2 \qquad (37)$$

Differentiating this gives

$$dx_t\, d^2x_t = dx_\theta\, d^2x_\theta + dx_\psi\, d^2x_\psi \qquad (38)$$

Applying Eq. 37 to Eq. 30 gives

$$d^2r/ds^2 = e^{-4\phi}\, \partial_1\phi(d\tau/ds)^2 - \partial_1\phi(dr/ds)^2 + (\partial_1\phi + 1/r)(dx_t/ds)^2 \qquad (39)$$

Multiplying Eq. 33 by (dx_θ/ds) gives

$$\begin{aligned} (dx_\theta/ds)(d^2x_\theta/ds^2) &= - (2\,\partial_1\phi + 1/r)(dr/ds)(dx_\theta/ds)^2 \\ &\quad + (1/r) \cot\theta\,(dx_\theta/ds)(dx_\psi/ds)^2 \qquad (40) \end{aligned}$$

Multiplying Eq. 36 by (dx_ψ/ds) gives

$$(dx_\psi/ds)(d^2x_\psi/ds^2) = - (2\partial_1\phi + 1/r]\,(dr/ds)\,(dx_\psi/ds)^2 - (1/r) \cot\theta\,(dx_\theta/ds)\,(dx_\psi/ds)^2 \qquad (41)$$

Equation 38 shows that the sum of Eqs 40, 41 is equal to $(dx_t/ds)(d^2x_t/ds^2)$. Adding Eqs. 40, 41 gives

$$\begin{aligned} (dx_t/ds)(d^2x_t/ds^2) &= - (2\partial_1\phi + 1/r)(dr/ds)(dx_\theta/ds)^2 + (1/r)\cot\theta\,(dx_\theta/ds)(dx_\psi/ds)^2 \\ &\quad - (2\partial_1\phi + 1/r]\,(dr/ds)\,(dx_\psi/ds)^2 - (1/r) \cot\theta\,(dx_\theta/ds)\,(dx_\psi/ds)^2 \qquad (42) \end{aligned}$$

In the right hand side, the second and fourth terms cancel. The equation simplifies to

$$(dx_t/ds)(d^2x_t/ds^2) = - (2\partial_1\phi + 1/r)(dr/ds)\{(dx_\theta/ds)^2 + (dx_\psi/ds)^2\} \qquad (43)$$

Applying Eq. 37 to this gives

$$(dx_t/ds)(d^2x_t/ds^2) = -(2\partial_1\phi + 1/r)(dr/ds)(dx_t/ds)^2 \tag{44}$$

Dividing by (dx_t/ds) gives

$$d^2x_t/ds^2 = -(2\partial_1\phi + 1/r)(dr/ds)(dx_t/ds) \tag{45}$$

Summary. The geodesic equations of the static Yilmaz theory for a spherically symmetric gravitational field reduce to the following three relations, which were obtained from Eqs 20, 39, and 45:

$$d^2\tau/ds^2 = 2\,\partial_1\phi\,(d\tau/ds)\,(dr/ds) \tag{46}$$

$$d^2r/ds^2 = e^{-4\phi}\,\partial_1\phi(d\tau/ds)^2 - \partial_1\phi(dr/ds)^2 + (\partial_1\phi + 1/r)(dx_t/ds)^2 \tag{47}$$

$$d^2x_t/ds^2 = -(2\partial_1\phi + 1/r)(dr/ds)(dx_t/ds) \tag{48}$$

3, Solution of Geodesic Equation for the Time Derivative

The geodesic equation for time given in Eq. 46 can be solved directly. Assume in Eq. 46 that the derivative $d\tau/ds$ has the form

$$d\tau/ds = C_1e^x \tag{49}$$

where C_1 is a constant. Differentiating this gives

$$d^2\tau/ds^2 = C_1e^x\,(dx/ds) = (d\tau/ds)\,(dx/ds) \tag{50}$$

In accordance with Eq. 49, the expression $(d\tau/ds)$ was substituted for C_1e^x. Applying this result to Eq. 46 gives

$$(dx/ds) = 2\,\partial_1\phi\,(dr/ds) \tag{51}$$

Since $\partial_1\phi$ is equal to $\partial\phi/\partial r$, this yields

$$dx = 2\,(d\phi/dr)\,dr = 2\,d\phi \tag{52}$$

Hence x is equal to 2ϕ, and so Eq. 49 becomes

$$d\tau/ds = C_1e^{2\phi} \tag{53}$$

When ϕ is zero, $d\tau/ds$ must be unity, and so the constant C_1 is unity. The solution for the geodesic equation for time is therefore

$$d\tau/ds = e^{2\phi} \tag{54}$$

Substitute this into Eq. 47 gives:

$$\begin{aligned} d^2r/ds^2 &= e^{-4\phi}\,\partial_1\phi\,(e^{2\phi})^2 - \partial_1\phi\,(dr/ds)^2 + (\partial_1\phi + 1/r)(dx_t/ds)^2 \\ &= \partial_1\phi - \partial_1\phi\,(dr/ds)^2 + (\partial_1\phi + 1/r)(dx_t/ds)^2 \\ &= (\partial\phi/\partial r)\,\{\,1 - (dr/ds)^2 + (dx_t/ds)^2\,\} + (1/r)(dx_t/ds)^2 \qquad (55) \end{aligned}$$

Hence, the geodesic equations simplify to the following

$$d\tau/ds = e^{2\phi} \qquad (56)$$

$$d^2r/ds^2 = (\partial\phi/\partial r)\{\,1 - (dr/ds)^2\,\} + \{\,(\partial\phi/\partial r) + (1/r)](dx_t/ds)^2 \qquad (57)$$

$$d^2x_t/ds^2 = -\,\{\,2(\partial\phi/\partial r) + 1/r\,\}\,(dr/ds)(dx_t/ds) \qquad (58)$$

These are the geodesic equations for the static Yilmaz theory when the gravitational field is spherically symmetric.

4, Geodesic Equations for Single-Star Model and Cosmology Model of Yilmaz Theory

We are considering two applications of the Yilmaz theory with spherically symmetric gravitational fields, which are: (1) the single star, and (2) the Yilmaz cosmology model. The values for 2ϕ and $\partial\phi/\partial r$ for these two cases are

Single star: $2\phi = 2m/r \;;\; \partial\phi/\partial r = -\,m/r^2$ (59)

Cosmology model: $2\phi = r^2/r_0^{\,2} \;;\; \partial\phi/\partial r = r/r_0^{\,2}$ (60)

Applying the values of Eqs. 59, 60 to the geodesic formulas of Eqs 56 to 58 gives the following geodesic formulas for the two cases.

Yilmaz Single-Star Model

$$d\tau/ds = e^{2m/r} \qquad (61)$$

$$d^2r/ds^2 = (1/r)(m/r)\{\,(dr/ds)^2 - 1\} + (1/r)[1 - (m/r)\}\,(dx_t/ds)^2 \qquad (62)$$

$$d^2x_t/ds^2 = (1/r)\{\,(2m/r) - 1\,\}\,(dr/ds)(dx_t/ds) \qquad (63)$$

Yilmaz Cosmology Model

$$d\tau/ds = \exp[(r/r_0)^2] \qquad (64)$$

$$d^2r/ds^2 = (r/r_0^{\,2})\{\,1 - (dr/ds)^2\,\} + (1/r)[1 + (r/r_0)^2](dx_t/ds)^2 \qquad (65)$$

$$d^2x_t/ds^2 \;=\; -\,(1/r)[1 + 2(r/r_0)^2\,](dr/ds)(dx_t/ds) \tag{66}$$

5, Calculation of Geodesic Equations for General Static Yilmaz Theory

This section derives general formulas for the geodesic equations of the general static Yilmaz theory expressed in rectangular coordinates. Combining Eqs, 1, 2 gives the following general formula for the geodesic equations:

$$d^2x^\alpha/ds^2 \;=\; -\;\Sigma_\mu\;\Sigma_\nu\;\Gamma_{\mu\nu}{}^{\alpha}\,(dx^\mu/ds)\,(dx^\nu/ds) \tag{67}$$

Let us separate the case for $\alpha = 0$ from those for $\alpha = j = 1, 2, 3$, to obtain:

$$d^2x^0/ds^2 \;=\; -\;\Sigma_\mu\;\Sigma_\nu\;\Gamma_{\mu\nu}{}^{0}\,(dx^\mu/ds)\,(dx^\nu/ds) \tag{68}$$

$$d^2x^j/ds^2 \;=\; -\;\Sigma_\mu\Sigma_\nu\,\Gamma_{\mu\nu}{}^{j}\,(dx^\mu/ds)\,(dx^\nu/ds) \quad j = 1, 2, 3 \tag{69}$$

In Appendix B, Eqs. B-24 to B-31 gave the following Christoffel symbol formulas for the static Yilmaz theory expressed in rectangular coordinates:

$$\Gamma_{\mu\nu}{}^{\alpha} = 0 \quad \text{if } \mu \neq \nu,\ \mu \neq \alpha,\ \nu \neq \alpha \qquad \text{(24 components)} \tag{70}$$

$$\Gamma_{00}{}^{0} \;=\; 0 \qquad \text{(1 component)} \tag{71}$$

$$\Gamma_{0k}{}^{k} \;=\; \Gamma_{k0}{}^{k} \;=\; \Gamma_{kk}{}^{0} \;=\; 0 \qquad \text{(9 components)} \tag{72}$$

$$\Gamma_{kk}{}^{k} \;=\; \partial_k\phi \qquad \text{(3 components)} \tag{73}$$

$$\Gamma_{k0}{}^{0} \;=\; \Gamma_{0k}{}^{0} \;=\; -\,\partial_k\phi \qquad \text{(6 components)} \tag{74}$$

$$\Gamma_{ki}{}^{i} \;=\; \Gamma_{ik}{}^{i} \;=\; \partial_k\phi \qquad (i \neq k) \qquad \text{(12 components)} \tag{75}$$

$$\Gamma_{ii}{}^{k} \;=\; -\,\partial_k\phi \qquad (i \neq k) \qquad \text{(6 components)} \tag{76}$$

$$\Gamma_{00}{}^{k} \;=\; -\,e^{-4\phi}\,\partial_k\phi \qquad \text{(3 components)} \tag{77}$$

The formula for d^2x^0/ds^2 in Eq. 68 can be expanded by setting μ equal to 0, k, and by setting ν equal to 0, m, where k, m have the values 1, 2, 3. The 16 terms of the formula are

$$\begin{aligned} d^2x^0/ds^2 \;&=\; -\;\Sigma_\mu\;\Sigma_\nu\;\Gamma_{\mu\nu}{}^{0}\,(dx^\mu/ds)\,(dx^\nu/ds) \\ &=\; -\;\Gamma_{00}{}^{0}\,(dx^0/ds)^2 - \;\Sigma_k\,\Sigma_m\;\Gamma_{km}{}^{0}\,(dx^k/ds)\,(dx^m/ds) \\ &\qquad -\;\Sigma_k\;\Gamma_{k0}{}^{0}\,(dx^k/ds)\,(dx^0/ds)\;-\;\Sigma_m\;\Gamma_{0m}{}^{0}\,(dx^0/ds)\,(dx^m/ds) \end{aligned} \tag{78}$$

Equation 71 shows that $\Gamma_{00}{}^{0} = 0$, and so the first term of Eq. 78 is zero. In accordance with Eq. 70, the Christoffel symbol $\Gamma_{km}{}^{0}$ is zero when neither k or m are zero unless k = m. Hence we can set m equal to k in the second term, and this reduces to s single summation over the index k. The last two terms of Eq. 78 are equal. Hence Eq. 78 reduces to

$$\begin{aligned} d^2x^0/ds^2 &= \Sigma_k\, \Gamma_{kk}{}^0\, (dx^k/ds)^2 - 2\, \Sigma_k\, \Gamma_{k0}{}^0\, (dx^k/ds)\, (dx^0/ds) \\ &= -\, 2\, \Sigma_k\, \Gamma_{k0}{}^0\, (dx^k/ds)\, (dx^0/ds) \qquad (79) \end{aligned}$$

Equation 72 shows that $\Gamma_{kk}{}^0 = 0$, and so the first term of Eq. 79 is zero, and the expression for d^2x^0/ds^2 simplifies to the form that is shown. .By Eq. 74, $\Gamma_{\mu 0}{}^0 = -\, \partial_\mu \phi$, for $\mu = k = 1, 2, 3$, and so this becomes

$$d^2x^0/ds^2 = 2\, (dx^0/ds)\;\; \Sigma_k\, \partial_k \phi\, (dx^k/ds) \qquad (80)$$

The formula for d^2x^j/ds^2 in Eq. 69 can be expanded as follows:

$$\begin{aligned} d^2x^j/ds^2 = &-\Gamma_{00}{}^j\, (dx^0/ds)(dx^0/ds) - \Gamma_{j0}{}^j\, (dx^j/ds)(dx^0/ds) - \Gamma_{0j}{}^j\, (dx^0/ds)(dx^j/ds) \\ &- \Gamma_{jj}{}^j\, (dx^j/ds)(dx^j/ds) - \Sigma_{k(k\neq j)}\, \Gamma_{jk}{}^j\, (dx^j/ds)\, (dx^k/ds) \\ &- \Sigma_{k(k\neq j)}\, \Gamma_{kj}{}^j\, (dx^k/ds)\, (dx^j/ds) - \Sigma_{k(k\neq j)}\, \Gamma_{kk}{}^j\, (dx^k/ds)\, (dx^k/ds) \qquad (81) \end{aligned}$$

The three summations are performed over the index k, but they omit the case where k is equal to j. This expansion has 10 terms. Equation 72 shows that terms 2 and 3 are zero, and Eq. 75 shows that terms 5, 6 are equal. Hence, Eq. 81 simplifies to

$$\begin{aligned} d^2x^j/ds^2 = &-\Gamma_{00}{}^j\, (dx^0/ds)(dx^0/ds) - \Gamma_{jj}{}^j\, (dx^j/ds)(dx^j/ds) \qquad (82) \\ &- 2\, \Sigma_{k(k\neq j)} \Gamma_{jk}{}^j\, (dx^j/ds)\, (dx^k/ds) - \Sigma_{k(k\neq j)} \Gamma_{kk}{}^j\, (dx^k/ds)\, (dx^k/ds) \end{aligned}$$

Apply Eq. 77 to term 1, Eq. 73 to term 2, Eq. 75 to term 3, and Eq. 76 to term 4. Equation 82 becomes

$$\begin{aligned} d^2x^j/ds^2 = &\; e^{-4\phi}\, \partial_j \phi\, (dx^0/ds)(dx^0/ds) - \partial_j \phi\, (dx^j/ds)(dx^j/ds) \qquad (83) \\ &- 2\, \Sigma_{k(k\neq j)}\;\; \partial_k \phi\, (dx^j/ds)(dx^k/ds) - \Sigma_{k(k\neq j)}\;\; \partial_j \phi\, (dx^k/ds)(dx^k/ds) \end{aligned}$$

This simplifies to

$$\begin{aligned} d^2x^j/ds^2 = &\; e^{-4\phi}\, \partial_j \phi (dx^0/ds)^2 - \partial_j \phi\, (dx^j/ds)^2 - \partial_j \phi\, \Sigma_{k(k\neq j)} (dx^k/ds)^2 \\ &- 2\, (dx^j/ds)\, \Sigma_{k(k\neq j)}\;\; \partial_k \phi\, (dx^k/ds) \qquad (84) \end{aligned}$$

Terms 2 and 3 can be combined into a single summation to give

$$d^2x^j/ds^2 = e^{-4\phi}\, \partial_j \phi\, (dx^0/ds)^2 - \partial_j \phi\, \Sigma_k\, (dx^k/ds)^2 - 2\, (dx^j/ds)\, \Sigma_{k(k\neq j)}\;\; \partial_k \phi\, (dx^k/ds) \qquad (85)$$

In Eqs. 80, 85, the variable x^0 can be replaced by τ to give the following, where the indices j, k have the values 1, 2, 3.

$$d^2\tau/ds^2 = 2\, (d\tau/ds)\;\; \Sigma_k\;\; \partial_k \phi\, (dx^k/ds) \qquad (86)$$

$$d^2x^j/ds^2 = e^{-4\phi}\,\partial_j\phi\,(d\tau/ds)^2 - \partial_j\phi\,\Sigma_{k(k\neq j)}\,(dx^k/ds)^2 - 2(dx^j/ds)\,\Sigma_k\,\partial_k\phi\,(dx^k/ds) \qquad (87)$$

6, **Simplified Geodesic Equations Applicable to Solar System**

The following discussion derives practical geodesic equations that can be used to calculate the orbits of bodies in our solar system. Equations 85, 87 are general formulas that can be applied to multi-body applications, such as the planetary orbits of our solar system. However, great simplification can be achieved by considering a more convenient formula to characterize the $d\tau/ds$ derivative. Equation 61 gave the following formula for this derivative that applies to a single star

$$d\tau/ds = e^{2\phi} = e^{2m/r} \qquad (88$$

In our solar system, the maximum value of $2m/r$ is $4.2\text{x}10^{-6}$, and so Eq. 88 can be approximated very accurately by

$$d\tau/ds = e^{2m/r} \cong 1 + 2m/r \qquad (89)$$

Since $2m/r$ can be no greater than $4.2\text{x}10^{-6}$, this derivative $d\tau/ds$ is extremely close to unity throughout our solar system. This shows that the minor changes in gravitational potential produced by the gravitational fields of the planets can have only infinitesimal effect on the $d\tau/ds$ derivative. Consequently, for general solar system calculations the $d\tau/ds$ derivative can be accurately calculated from Eq. 88, which considers only the gravitational field of the sun

Applying Eq. 88 to the first term of Eq. 87 gives

$$d^2x^j/ds^2 = \partial_j\phi - \partial_j\phi\,\Sigma_k\,(dx^k/ds)^2 - 2(dx^j/ds)\,\Sigma_{k(k\neq j)}\,\partial_k\phi(dx^k/ds) \qquad (90)$$

Solving Eq. 88 for ds gives

$$ds = e^{-2\phi}\,d\tau = e^{-2\phi}\,c\,dt \qquad (91)$$

By applying this to Eq. 90, the derivatives in the equation can be expressed as follows in terms of time:

$$d^2x^j/dt^2 = e^{-4\phi}\,c^2\,\partial_j\phi - \partial_j\phi\,\Sigma_k\,(dx^k/dt)^2 - 2(dx^j/dt)\,\Sigma_{k(k\neq j)}\,\partial_k\phi\,(dx^k/dt) \qquad (92)$$

The expression dx^j/dt is the derivative in the x^j direction, which is the velocity in the x^j direction. The expression d^2x^j/dt^2 is the second derivative in the x^j direction, which is the acceleration in the x^j direction. Equation 92 yields the following formulas for the accelerations A_x, A_y, A_z in the x, y, and z directions, where V_x, V_y, V_z are the x, y, z velocities:

$$A_x = (\partial\phi/\partial x)(c^2e^{-4\phi} - V^2) - 2(\partial\phi/\partial y)V_xV_y - 2(\partial\phi/\partial z)V_xV_z \qquad (93)$$

$$A_y = (\partial\phi/\partial y)(c^2e^{-4\phi} - V^2) - 2(\partial\phi/\partial x)V_yV_x - 2(\partial\phi/\partial z)V_yV_z \qquad (94)$$

$$A_z = (\partial\phi/\partial z)(c^2e^{-4\phi} - V^2) - 2(\partial\phi/\partial x)V_zV_x - 2(\partial\phi/\partial y)V_zV_y \qquad (95)$$

The variable V is the absolute value of velocity, given by

$$V^2 = V_x{}^2 + V_y{}^2 + V_z{}^2 \qquad (96)$$

7 **Geodesic Equations for Yilmaz Single Star Model in Terms of Time**

It is desirable to express the geodesic equations for the Yilmaz single-star model in terms of normalized time instead of the variable s. Equation 61 gives the following:

$$d\tau/ds = e^{2m/r} \qquad \{(61)\}$$

From this one can express the derivatives along the geodesic trajectory of r and x_t as follows:

$$dr/ds = (dr/d\tau)(d\tau/ds) = (dr/d\tau)e^{2m/r} \qquad (97)$$

$$dx_t/ds = (dx_t/d\tau)(d\tau/ds) = (dx_t/d\tau)e^{2m/r} \qquad (98)$$

The second derivative of x_t is

$$\begin{aligned} d^2x_t/ds^2 &= (d/ds)(dx_t/ds) = (d\tau/ds)[(d/d\tau)(dx_t/ds)] \\ &= (d\tau/ds)[(d/d\tau)(dx_t/d\tau)e^{2m/r}] \\ &= (d\tau/ds)\{ e^{2m/r}(d/d\tau)(dx_t/d\tau) + (dx_t/d\tau)(d/d\tau)e^{2m/r} \} \\ &= e^{2m/r}\{ e^{2m/r}(d^2x_t/d\tau^2) + (dx_t/d\tau)e^{2m/r}[-2(m/r^2)(dr/d\tau)] \} \\ &= e^{4m/r} [(d^2x_t/d\tau^2) - 2(m/r^2)(dx_t/d\tau)(dr/d\tau)] \qquad (99) \end{aligned}$$

Similarly, the second derivative of r is

$$d^2r/ds^2 = e^{4m/r} [(d^2r/d\tau^2) - 2(m/r^2)(dr/d\tau)(dr/d\tau)] \qquad (100)$$

Equations 62, 63 gave the following geodesic equations:

$$d^2r/ds^2 = (1/r)(m/r)\{ (dr/ds)^2 - 1\} + (1/r)[1 - (m/r)\} (dx_t/ds)^2 \qquad \{(62)\}$$

$$d^2x_t/ds^2 = (1/r)\{ (2m/r) - 1 \} (dr/ds)(dx_t/ds) \qquad \{(63)\}$$

Substitute Eqs 97, 98 into the right hand sides of Eqs. 62, 63. This gives

$$d^2r/ds^2 = (1/r)(m/r)\{ (dr/d\tau)^2 e^{4m/r} - 1\} + (1/r)[1 - (m/r)\} (dx_t/d\tau)^2 e^{4m/r} \qquad (101)$$

$$d^2x_t/ds^2 = (1/r)\{ (2m/r) - 1 \} (dr/d\tau)(dx_t/d\tau) e^{4m/r} \qquad (102)$$

Substitute Eq. 100 into the left side of Eq. 101:

$$e^{4m/r}\,[(d^2r/d\tau^2) - 2(m/r^2)(dr/d\tau)^2] = (1/r)(m/r)\{(dr/d\tau)^2 e^{4m/r} - 1\} + (1/r)[1 - (m/r)\}(dx_t/d\tau)^2 e^{4m/r} \qquad (103)$$

Multiply both sides for $e^{-4m/r}$, and solve for $(d^2r/d\tau^2)$:

$$d^2r/d\tau^2 = 2(m/r^2)(dr/d\tau)^2 + (1/r)(m/r)\{(dr/d\tau)^2 - e^{-4m/r}\} + (1/r)[1 - (m/r)\}(dx_t/d\tau)^2 \qquad (104)$$

Substitute Eq. 99 into the left side of Eq. 102:

$$e^{4m/r}\,[(d^2x_t/d\tau^2) - 2(m/r^2)(dx_t/d\tau)(dr/d\tau)] = (1/r)\{ (2m/r) - 1 \}(dr/d\tau)(dx_t/d\tau)\, e^{4m/r} \qquad (105)$$

Multiply both sides for $e^{-4m/r}$, and solve for $(d^2x_t/d\tau^2)$:

$$d^2x_t/d\tau^2 = 2(m/r^2)(dx_t/d\tau)(dr/d\tau) + (1/r)\{ (2m/r) - 1 \}(dr/d\tau)(dx_t/d\tau) \qquad (106)$$

Equations 104, 106 simplify to the following:

$$r(d^2r/d\tau^2) = 3(m/r)(dr/d\tau)^2 - (m/r)e^{-4m/r} + [1 - (m/r)](dx_t/d\tau)^2 \qquad (107)$$

$$r(d^2x_t/d\tau^2) = \{(4m/r) - 1 \}(dr/d\tau)(dx_t/d\tau) \qquad (108)$$

References

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