5,F Addendum Appendix F
Stress-Energy Tensors of General Yilmaz Theory

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This Appendix derives formulas for the stress-energy tensors $t_\mu{}^\nu$, $\tau_\mu{}^\nu$ of the general time-varying Yilmaz theory. The Primary analysis is presented in Section 1; Sections 2 and 3 derive formulas that are used in Section 1

Based on formulas given by Pauli [8], equations are derived in Section 2 for the pseudo-tensors denoted $u_\mu{}^\nu$ and $U_\mu{}^\nu$ by Yilmaz. In Section 1, these equations are expressed in harmonic coordinates, and gauge conditions are applied. The application of harmonic coordinates converts the pseudo-tensors to true tensors, which are directly related to the stress-energy tensors $-t_\mu{}^\nu$, $\tau_\mu{}^\nu$ of the Yilmaz theory. In harmonic coordinates, the pseudo tensor ($u_\mu{}^\nu$) is equal to the true tensor ($-t_\mu{}^\nu$), and the pseudo-tensor ($U_\mu{}^\nu$) is equal to the true tensor ($\tau_\mu{}^\nu$). Tensor $t_\mu{}^\nu$ is the stress-energy tensor for the gravitational field, and $\tau_\mu{}^\nu$ is the stress-energy tensor for matter.

When indices in a tensor equation are repeated, with one index a subscript and the other a superscript, the expression is to be summed over the four values (0, 1, 2, 3) of the index. This website normally indicates an implied summation by a Σ sign. However in this appendix the Σ symbols for implied summations are omitted.

1, Formulas for Stress-Energy Tensors

1,1 Application of Gauge Conditions and Harmonic Coordinates

The formulas for $U_\mu{}^\nu$ and $u_\mu{}^\nu$ can be simplified by applying gauge conditions and harmonic coordinates. The following gauge conditions apply:

$$\partial_\nu \phi_\mu{}^\nu = 0 \qquad (1\text{-}1)$$

$$\partial^\mu \phi_\mu{}^\nu = 0 \qquad (1\text{-}2)$$

Yilmaz [6] (App. B, p. 959), shows that the following identities hold with harmonic coordinates:

$$\partial_\nu \mathbf{g}^{\mu\nu} = 0 \qquad (1\text{-}3)$$

$$\partial^\nu \mathbf{g}_{\mu\nu} = 0 \qquad (1\text{-}4)$$

Bold letters denote the following functions, which are called *tensor densities*:

$$\mathbf{g}^{\mu\nu} = \sqrt{[-g]}\, g^{\mu\nu} \quad , \quad \mathbf{g}_{\mu\nu} = g_{\mu\nu}/\sqrt{[-g]} \qquad (1\text{-}5)$$

(Tensor densities are traditionally represented by Old German script, but that is difficult to duplicate in a website.) The harmonic coordinate condition, defined by these equations, greatly simplifies the analysis. It permits a wave to propagate in a single direction, without having a compensating wave in the reverse direction. Hence it allows energy to be radiated. These conditions can apply to electromagnetic waves and to gravitational waves.

The following partial-derivative definitions are used in this appendix:

$$\partial_\alpha f = \partial f/\partial x^\alpha \quad (1\text{-}6)$$

$$\partial^\alpha f = g^{\alpha\beta}\,\partial_\beta f = \Sigma_\beta\, g^{\alpha\beta}\,\partial_\beta f \quad (1\text{-}7)$$

1,2 Simplification of Formula for Pseudo-Tensor $U_\mu{}^\nu$

Equation 2-13 will give the following formula for the pseudo-tensor $U_\mu{}^\nu$:

$$U_\mu{}^\nu = (1/4\sqrt{[-g]})\,\partial_\alpha\{\mathbf{g}^{\alpha\lambda}\,\mathbf{g}^{\nu\rho}(\partial_\rho\,\mathbf{g}_{\mu\lambda} - \partial_\lambda\,\mathbf{g}_{\mu\rho}) + \delta_\mu{}^\nu\,\partial_\beta\,\mathbf{g}^{\beta\alpha} - \delta_\mu{}^\alpha\,\partial_\beta\,\mathbf{g}^{\beta\nu}\} \quad (1\text{-}8)$$

When the harmonic coordinate condition of Eq. 1-3 is applied, the last two terms of this expression for $U_\mu{}^\nu$ become zero, and the equation simplifies to

$$4\sqrt{[-g]}U_\mu{}^\nu = \partial_\alpha[\mathbf{g}^{\nu\rho}\{\mathbf{g}^{\alpha\lambda}\,\partial_\rho\mathbf{g}_{\mu\lambda}\}] - \partial_\alpha[\mathbf{g}^{\alpha\lambda}\{\mathbf{g}^{\nu\rho}\,\partial_\lambda\mathbf{g}_{\mu\rho}\}] \quad (1\text{-}9)$$

Equation 3-38 will show that the expressions within the braces { } are equal to

$$\{\,\mathbf{g}^{\alpha\lambda}\,\partial_\rho\,\mathbf{g}_{\mu\lambda}\,\} = -4\,\partial_\rho\,\phi_\mu{}^\alpha \quad (1\text{-}10)$$

$$\{\,\mathbf{g}^{\nu\rho}\,\partial_\lambda\,\mathbf{g}_{\mu\rho}\,\} = -4\,\partial_\lambda\,\phi_\mu{}^\nu \quad (1\text{-}11)$$

Substituting Eqs 1-10, 1-11 into Eq. 1-9 gives

$$\sqrt{[-g]}U_\mu{}^\nu = -\,\partial_\alpha\,[\mathbf{g}^{\nu\rho}\,\partial_\rho\,\phi_\mu{}^\alpha] + \partial_\alpha\,[\mathbf{g}^{\alpha\lambda}\,\partial_\lambda\,\phi_\mu{}^\nu\,] \quad (1\text{-}12)$$

Reverse terms, and divide by $\sqrt{[-g]}$. This gives

$$U_\mu{}^\nu = (1/\sqrt{[-g]})\partial_\alpha\,[\,\mathbf{g}^{\alpha\lambda}\partial_\lambda\,\phi_\mu{}^\nu\,] - (1/\sqrt{[-g]})\,\partial_\alpha\,[\mathbf{g}^{\nu\rho}\,\partial_\rho\,\phi_\mu{}^\alpha] \quad (1\text{-}13)$$

The first term of Eq. 1-13 is the d'Alembertian of $\phi_\mu{}^\nu$, which is denoted $\Box^2\phi_\mu{}^\nu$, and is defined by

$$\Box^2\phi_\mu{}^\nu = (1/\sqrt{[-g]})\,\partial_\alpha[\,\mathbf{g}^{\alpha\lambda}\,\partial_\lambda\,\phi_\mu{}^\nu\,] \quad (1\text{-}14)$$

The second term of Eq. 1-13 can be modified by applying the definition of $\mathbf{g}^{\mu\nu}$ in Eq. 1-5 and the following definition derived from Eq. 1-7:

$$\partial^\nu\phi_\mu{}^\alpha = g^{\nu\rho}\,\partial_\rho\phi_\mu{}^\alpha \quad (1\text{-}15)$$

The second term of Eq. 1-13 becomes

$$- (1/\sqrt{[-g]})\, \partial_\alpha[\, \sqrt{[-g]})\, g^{\nu\rho}\, \partial_\rho \phi_\mu{}^\alpha\,] \;=\; - (1/\sqrt{[-g]})\, \partial_\alpha[\sqrt{[-g]})\, \partial^{\,\nu} \phi_\mu{}^\alpha\,] \qquad (1\text{-}16)$$

Applying Eq. 1-14, 1-15 to Eq. 1-13 gives

$$U_\mu{}^\nu \;=\; \Box\Box^2 \phi_\mu{}^\nu \; - (1/\sqrt{[-g]})\, \partial_\alpha[\sqrt{[-g]}\, \partial^\nu \phi_\mu{}^\alpha\,] \qquad (1\text{-}17)$$

It can be shown that this is a tensor. As will be explained in Section 2, $U_\mu{}^\nu$ is equal to the sum $(\tau_\mu{}^\nu + z_\mu{}^\nu)$, where $z_\mu{}^\nu$ is a non-tensor. By expressing $U_\mu{}^\nu$ in harmonic coordinates, the non-tensor component becomes zero, and so $U_\mu{}^\nu$ becomes the stress-energy tensor for matter $\tau_\mu{}^\nu$. Setting $U_\mu{}^\nu$ equal to $\tau_\mu{}^\nu$ in Eq. 17 gives

$$\tau_\mu{}^\nu \;= \Box\Box^2 \phi_\mu{}^\nu \; - (1/\sqrt{[-g]})\, \partial_\alpha[\sqrt{[-g]}\, \partial^\nu \phi_\mu{}^\alpha\,] \qquad (1\text{-}18)$$

This agrees with the expression for $\tau_\mu{}^\nu$ given by Yilmaz [7] (p. 498).

The expression for $\Box^2 \phi_\mu{}^\nu$ in Eq. 1-14 can be simplified as follows by replacing $g^{\alpha\lambda}$ by the definition of Eq. 1-8, and then by replacing $\sqrt{[-g]}$ by $e^{2\phi}$ in accordance with Eq. 3-29:

$$\Box\Box^2 \phi_\mu{}^\nu \;=\; (1/\sqrt{[-g]}) \partial_\alpha[\mathbf{g}^{\boldsymbol{\alpha\lambda}}\, \partial_\lambda \phi_\mu{}^\nu] = \; e^{-2\phi} \partial_\alpha[\sqrt{[-g]} g^{\alpha\lambda}\, \partial_\lambda \phi_\mu{}^\nu]$$

$$= \; e^{-2\phi} \partial_\alpha[e^{2\phi}\, g^{\alpha\lambda} \partial_\lambda \phi_\mu{}^\nu] = \; e^{-2\phi} \partial_\alpha[e^{2\phi}\, \partial^\alpha \phi_\mu{}^\nu] \qquad (1\text{-}19)$$

The definition for $\partial^\alpha \phi_\mu{}^\nu$ given in Eq. 1-15 was applied in the last term. Substitute this into Eq. 1-18, and replace $\sqrt{[-g]}$ by $e^{2\phi}$ in the second term. Then expand the derivatives. This gives

$$\tau_\mu{}^\nu \;=\; e^{-2\phi}\, \partial_\alpha[e^{2\phi}\, \partial^\alpha \phi_\mu{}^\nu] - e^{-2\phi}\, \partial_\alpha[e^{2\phi}\, \partial^\nu \phi_\mu{}^\alpha\,]$$

$$= \; e^{-2\phi}\{e^{2\phi} \partial_\alpha \partial^\alpha \phi_\mu{}^\nu + \partial^\alpha \phi_\mu{}^\nu (e^{2\phi}\, 2\partial_\alpha \phi)\} - \; e^{-2\phi}\{e^{2\phi}\, \partial_\alpha \partial^\nu \phi_\mu{}^\alpha + \partial^\nu \phi_\mu{}^\alpha (e^{2\phi}\, 2\partial_\alpha \phi)\}$$

$$= \; \partial_\alpha \partial^\alpha \phi_\mu{}^\nu - \partial_\alpha \partial^\nu \phi_\mu{}^\alpha + 2\partial_\alpha \phi\, \{\partial^\alpha \phi_\mu{}^\nu \; - \partial^\nu \phi_\mu{}^\alpha\} \qquad (1\text{-}20)$$

This simplifies to

$$\tau_\mu{}^\nu = \; \partial_\alpha\{\partial^\alpha \phi_\mu{}^\nu - \partial^\nu \phi_\mu{}^\alpha\} + 2\partial_\alpha \phi\{\partial^\alpha \phi_\mu{}^\nu \; - \partial^\nu \phi_\mu\} \qquad (1\text{-}21)$$

1,3 Simplification of Formulas for $W_\mu{}^\nu$ and $u_\mu{}^\nu$

As will be explained in Section 2, the pseudo-tensor $u_\mu{}^\nu$ is expressed in terms of the pseudo-tensor $W_\mu{}^\nu$, which is given as follows in Eq. 2-21:

$$W_\mu{}^\nu \;=\; (1/8) g^{\nu\lambda}\, \partial_\lambda \mathbf{g}_{\boldsymbol{\alpha\beta}}\, \partial_\mu \mathbf{g}^{\boldsymbol{\alpha\beta}} - \tfrac{1}{4}\, g^{\nu\lambda}\, \partial_\lambda(\sqrt{[-g]})\, \partial_\mu(1/\sqrt{[-g]}) - \tfrac{1}{4}\, g^{\nu\lambda}\, \partial_\alpha \mathbf{g}_{\boldsymbol{\lambda\beta}}\, \partial_\mu \mathbf{g}^{\boldsymbol{\alpha\beta}} \qquad (1\text{-}22)$$

The third (last) term is denoted $[W_\mu{}^\nu]_3$, and can be expressed as follows by differentiating by parts:

$$[W_\mu{}^\nu]_3 = -\tfrac{1}{4} g^{\nu\lambda}\, \partial_\alpha\, \mathbf{g}_{\lambda\beta}\, \partial_\mu\, \mathbf{g}^{\alpha\beta} = -\tfrac{1}{4} g^{\nu\lambda}\, [\partial_\alpha[g_{\lambda\beta}\, \partial_\mu g^{\alpha\beta}] - g_{\lambda\beta}\, \partial_\alpha\partial_\mu g^{\alpha\beta}]$$

$$= -\tfrac{1}{4} g^{\nu\lambda}\, [\partial_\alpha[\mathbf{g}_{\lambda\beta}\, \partial_\mu \mathbf{g}^{\alpha\beta}] - \mathbf{g}_{\lambda\beta}\partial_\mu\{\partial_\alpha \mathbf{g}^{\alpha\beta}\}] \qquad (1\text{-}23)$$

In accordance with the harmonic coordinate condition of Eq. 1-3, the expression within the braces { } is zero. This term reduces to

$$[W_\mu{}^\nu]_3 = -\tfrac{1}{4} g^{\nu\lambda}\, \partial_\alpha\{\mathbf{g}_{\lambda\beta}\, \partial_\mu \mathbf{g}^{\alpha\beta}\} = -\tfrac{1}{4} g^{\nu\lambda}\partial_\alpha[4\partial_\mu\phi_\lambda{}^\alpha] = -g^{\nu\lambda}\, \partial_\mu[\partial_\alpha\phi_\lambda{}^\alpha] \quad (1\text{-}24)$$

The expression within the braces was replaced with the relation to be given in Eq. 3-40. In accordance with the gauge condition of Eq. 1-1, the expression $\partial_\alpha\phi_\lambda{}^\alpha$ in the last expression of Eq. 1-24 is zero, and so this third term $[W_\mu{}^\nu]_3$ of Eq. 1-22 is zero.

Denote the second term of $W_\mu{}^\nu$ in Eq. 1-22 as $[W_\mu{}^\nu]_2$. In accordance with Eq. 3-29, set $\sqrt{[-g]})$ equal to $e^{2\phi}$. This gives

$$[W_\mu{}^\nu]_2 = -\tfrac{1}{4} g^{\nu\lambda}\partial_\lambda(\sqrt{[-g]})\partial_\mu(1/\sqrt{[-g]}) \qquad = -\tfrac{1}{4} g^{\nu\lambda}\{\partial_\lambda e^{2\phi}\}(\partial_\mu e^{-2\phi})$$

$$= -\tfrac{1}{4} g^{\nu\lambda}\{2e^{2\phi}\partial_\lambda\phi\}[-2e^{-2\phi}\partial_\mu\phi] = g^{\nu\lambda}\partial_\lambda\phi\partial_\mu\phi = \partial_\mu\phi\partial^\nu\phi \qquad (1\text{-}25)$$

Equation 3-42 will give the following relation:

$$\partial_\lambda\, \mathbf{g}_{\alpha\rho}\, \partial_\mu\, \mathbf{g}^{\alpha\rho} = -16\, \partial_\lambda\, \phi_\alpha{}^\beta\, \partial_\mu\, \phi_\beta{}^\alpha \qquad (1\text{-}26)$$

Substitute this into the first term of Eq. 1-22, and denote this first term as $[W_\mu{}^\nu]_1$. This gives

$$[W_\mu{}^\nu]_1 = -2\, \{ g^{\nu\lambda}\, \partial_\lambda\, \phi_\alpha{}^\beta \}\, \partial_\mu\, \phi_\beta{}^\alpha \qquad (1\text{-}27)$$

By applying the Eq. 1-6 definition to the expression within the { } braces, Eq. 1-27 simplifies to

$$[W_\mu{}^\nu]_1 = -2\, \partial^\nu\, \phi_\alpha{}^\beta\, \partial_\mu\, \phi_\beta{}^\alpha = -2\, \partial_\mu\, \phi_\beta{}^\alpha\, \partial^\nu\, \phi_\alpha{}^\beta \qquad (1\text{-}28)$$

The total of $W_\mu{}^\nu$ is the sum of Eqs 1-28, 1-25, which is

$$W_\mu{}^\nu = -2\, \partial_\mu\, \phi_\beta{}^\alpha\, \partial^\nu\, \phi_\alpha{}^\beta + \partial_\mu\phi\, \partial^\nu\phi \qquad (1\text{-}29)$$

The trace of $W_\mu{}^\nu$, which is the sum of its diagonal components, is denoted W and is equal to

$$W = W_0{}^0 + W_1{}^1 + W_2{}^2 + W_3{}^3 = -2\, \partial_\lambda\, \phi_\beta{}^\alpha\, \partial^\lambda\, \phi_\alpha{}^\beta + \partial_\lambda\, \phi\, \partial^\lambda\, \phi \qquad (1\text{-}30)$$

By Eq. 2-8, $u_\mu{}^\nu$ is equal to

$$u_\mu{}^\nu = - W_\mu{}^\nu + ½ \delta_\mu{}^\nu W \qquad (1\text{-}31)$$

Applying Eqs 1-29, 1-30 to Eq. 1-31 gives

$$u_\mu{}^\nu = 2 \partial_\mu \phi_\beta{}^\alpha \partial^\nu \phi_\alpha{}^\beta - \partial_\mu\phi \partial^\nu\phi - \delta_\mu{}^\nu\{\partial_\lambda \phi_\beta{}^\alpha \partial^\lambda \phi_\alpha{}^\beta - ½ \partial_\lambda \phi \partial^\lambda \phi \} \qquad (1\text{-}32)$$

Section 2 shows that $u_\mu{}^\nu$ is equal to $(-t_\mu{}^\nu + z_\mu{}^\nu)$, where $z_\mu{}^\nu$ is a non-tensor. It can be shown that Eq. 1-46 is a tensor, and so the non-tensor component of $u_\mu{}^\nu$ must be zero. Hence, Eq. 1-32 must be equal to $-t_\mu{}^\nu$. Taking the negative of Eq. 1-32 gives the following formula for $t_\mu{}^\nu$:

$$t_\mu{}^\nu = - u_\mu{}^\nu = - 2 \partial_\mu \phi_\beta{}^\alpha \partial^\nu \phi_\alpha{}^\beta + \partial_\mu \phi \partial^\nu \phi + \delta_\mu{}^\nu\{\partial_\lambda \phi_\beta{}^\alpha \partial^\lambda \phi_\alpha{}^\beta - ½ \partial_\lambda \phi \partial^\lambda \phi\} \qquad (1\text{-}33)$$

1,4 Summary of Derivation

The formulas for the stress-energy tensors $t_\mu{}^\nu$, $\tau_\mu{}^\nu$ of the Yilmaz theory are given in Eqs 1-33, 1-21, and are repeated as follows:

$$t_\mu{}^\nu = - 2 \partial_\mu \phi_\beta{}^\alpha \partial^\nu \phi_\alpha{}^\beta + \partial_\mu \phi \partial^\nu \phi + \delta_\mu{}^\nu\{\partial_\lambda \phi_\beta{}^\alpha \partial^\lambda \phi_\alpha{}^\beta - ½ \partial_\lambda \phi \partial^\lambda \phi \qquad (1\text{-}34)$$

$$\tau_\mu{}^\nu = \partial_\alpha\{\partial^\alpha \phi_\mu{}^\nu - \partial^\nu \phi_\mu{}^\alpha\} + 2 \partial_\alpha \phi \{\partial^\alpha \phi_\mu{}^\nu - \partial^\nu \phi_\mu{}^\alpha \} \qquad (1\text{-}35)$$

As was shown in Eq. 1-18, $\tau_\mu{}^\nu$ can also be expressed as:

$$\tau_\mu{}^\nu = \Box^2 \phi_\mu{}^\nu - (1/\sqrt{[-g]}) \partial_\alpha[\sqrt{[-g]} \partial^\nu \phi_\mu{}^\alpha] \qquad (1\text{-}36)$$

In Eq. 1-34 the dummy index ρ was replaced by λ in the third term. Equations 1-34, 1-36 are the same as the formulas for $t_\mu{}^\nu$, $\tau_\mu{}^\nu$ given by Yilmaz [7] (p. 498).

2, Formulas for Yilmaz Pseudo-Tensors $u_\mu{}^\nu$ and $U_\mu{}^\nu$

This section uses equations given by Pauli [8] to derive formulas for the pseudo tensors $U_\mu{}^\nu$ and $u_\mu{}^\nu$ that are used by Yilmaz to describe the general time-varying Yilmaz theory. These pseudo tensors are related to the true stress-energy tensors $\tau_\mu{}^\nu$ and $t_\mu{}^\nu$.

2,1 Pauli Pseudo-Tensors $t_\mu{}^\nu$ and $U_\mu{}^\nu$

Let us refer to the book by Pauli [8], *Theory of Relativity*. The body of Pauli's book was first published in 1921 as a part of the *Mathematical Encyclopedia*. In 1956, this material was republished by Pauli without change. To bring the material up to date, Pauli added to it a set of *Supplementary Notes*, which give selected information on later developments of relativity theory, along with Pauli's personal views on some controversial questions.

In his book, Pauli uses the variables $U_\mu{}^\nu$ and $t_\mu{}^\nu$, which differ from variables with the same symbols used by Yilmaz. To avoid confusion, this website indicates these Pauli symbols with strike-through marks. Hence, the Pauli symbols $U_\mu{}^\nu$ and $t_\mu{}^\nu$ are replaced by $\not{U}_\mu{}^\nu$ and $\not{t}_\mu{}^\nu$.

This website uses Greek letters (such as μ, ν) to represent 4-dimensional indices, whereas Pauli uses Roman letters (such as i, k) for this purpose. Hence Pauli's Roman indices i, k are replaced with the Greek indices μ, ν.

As shown by Pauli [8] (p. 162, Eq. 406), the variables $\text{Ʉ}_\mu{}^\nu$ and $\text{ŧ}_\mu{}^\nu$ are related by

$$\text{Ʉ}_\mu{}^\nu = - \kappa\, \text{ŧ}_\mu{}^\nu = - 8\pi\, \text{ŧ}_\mu{}^\nu \qquad (2\text{-}1)$$

The constant κ is equal to 8π in normalized relativistic units. The variables $\text{Ʉ}_\mu{}^\nu$ and $\text{ŧ}_\mu{}^\nu$ are pseudo-tensors, which means that they have the appearance of tensors, but they do not transform like true tensors. As is shown elsewhere, the pseudo-tensor $\text{ŧ}_\mu{}^\nu$ was considered by Einstein to represent the "energy components of the gravitational field".

A *tensor density* is formed by multiplying a tensor by $\sqrt{[-g]}$, where g is the determinant of the metric tensor $g_{\mu\nu}$. For the general time-varying Yilmaz theory, $\sqrt{[-g]}$ is equal to $e^{2\phi}$. Pauli denotes tensor densities by old German script, but we use bold letters instead. Pauli defines the following tensor densities:

$$\mathbf{G}_\mu{}^\nu = \sqrt{[-g]}\, G_\mu{}^\nu \qquad (2\text{-}2)$$

$$\mathbf{U}_\mu{}^\nu = \sqrt{[-g]}\, \text{Ʉ}_\mu{}^\nu \qquad (2\text{-}3)$$

The tensor $G_\mu{}^\nu$ is the Einstein tensor. Pauli (p. 70, Eq 185) gives the following formula for $\mathbf{U}_\mu{}^\nu$:

$$2\, \mathbf{U}_\mu{}^\nu = \Gamma_{ar}{}^r\, \partial_\mu\{ g^{a\nu}\sqrt{[-g]}\} - \Gamma_{rs}{}^\nu\, \partial_\mu\{g^{rs}\sqrt{[-g]}\} - \mathbf{G}\, \delta_\mu{}^\nu \qquad (2\text{-}4)$$

The symbol **G** is defined by Pauli as follows on page 68, Eq 178:

$$\mathbf{G} = \sqrt{[-g]}\, g^{ik} (\Gamma_{is}{}^r\, \Gamma_{kr}{}^s - \Gamma_{ik}{}^r\, \Gamma_{rs}{}^s) \qquad (2\text{-}5)$$

Pauli (p. 215, Eq II) gives the following identity, where the Pauli indices i, k, l have been replaced by μ, ν, α:

$$\mathbf{U}_\mu{}^\nu + \mathbf{G}_\mu{}^\nu \equiv \partial_\alpha\, \mathbf{B}_\mu{}^{\nu\alpha} \qquad (2\text{-}6)$$

Pauli (p. 216, Eq. 1) gives the following formula for $\mathbf{B}_\mu{}^{\nu\alpha}$, where the Pauli indices i, k, l have been replaced by μ, ν, α, and the indices r, s have been replaced by ρ, λ.

$$2\mathbf{B}_\mu{}^{\nu\alpha} = \sqrt{[-g]}\{\delta_\mu{}^\nu(g^{\rho\lambda}\, \Gamma_{\rho\lambda}{}^\alpha - g^{\alpha\rho}\, \Gamma_{\rho\lambda}{}^\lambda) + \delta_\mu{}^\alpha(g^{\nu\rho}\, \Gamma_{\rho\lambda}{}^\lambda - g^{\rho\lambda}\, \Gamma_{\rho\lambda}{}^\nu) + (g^{\alpha\rho}\, \Gamma_{\mu\rho}{}^\nu - g^{\nu\rho}\, \Gamma_{\mu\rho}{}^\alpha)\} \qquad (2\text{-}7)$$

This formula was first derived by P. Freud in 1935 [9].

2,2 Calculating Formulas for Yilmaz Pseudo-Tensors $U_\mu^{\ \nu}$ and $u_\mu^{\ \nu}$.

Yilmaz has defined his pseudo-tensor $U_\mu^{\ \nu}$ and $u_\mu^{\ \nu}$ as follows:

$$U_\mu^{\ \nu} = -\tfrac{1}{2} G_\mu^{\ \nu} - \tfrac{1}{2} \not{U}_\mu^{\ \nu} \qquad (2\text{-}8)$$

$$u_\mu^{\ \nu} = -\tfrac{1}{2} \not{U}_\mu^{\ \nu} \qquad (2\text{-}9)$$

The tensor densities of the variables of Eq. 2-8 are related by

$$-2\sqrt{[-g]}U_\mu^{\ \nu} = \sqrt{[-g]}(G_\mu^{\ \nu} + \not{U}_\mu^{\ \nu}) = \mathbf{G}_\mu^{\ \nu} + \mathbf{U}_\mu^{\ \nu} \qquad (2\text{-}10)$$

Combining Eqs 2-6, 2-10 gives

$$4\sqrt{[-g]}U_\mu^{\ \nu} = -2\,\partial_\alpha \mathbf{B}_\mu^{\ \nu\alpha} \qquad (2\text{-}11)$$

Applying Eq. 2-11 to Eq. 2-7 gives

$$4\sqrt{[-g]}U_\mu^{\ \nu} = -\partial_\alpha\{\sqrt{[-g]}[\delta_\mu^{\ \nu}(g^{\rho\lambda}\,\Gamma_{\rho\beta}^{\ \ \alpha} - g^{\alpha\rho}\,\Gamma_{\rho\lambda}^{\ \ \lambda}) + \delta_\mu^{\ \alpha}(g^{\nu\rho}\,\Gamma_{\rho\lambda}^{\ \ \lambda} - g^{\rho\lambda}\,\Gamma_{\rho\lambda}^{\ \ \nu})$$
$$+ (g^{\alpha\rho}\,\Gamma_{\mu\rho}^{\ \ \nu} - g^{\nu\alpha}\,\Gamma_{\mu\rho}^{\ \ \alpha})]\}$$
$$= \partial_\alpha\{\sqrt{[-g]}[\delta_\mu^{\ \nu}(g^{\alpha\rho}\,\Gamma_{\rho\lambda}^{\ \ \lambda} - g^{\rho\lambda}\,\Gamma_{\rho\beta}^{\ \ \alpha}) + \delta_\mu^{\ \alpha}(g^{\rho\lambda}\,\Gamma_{\rho\lambda}^{\ \ \nu} - g^{\nu\rho}\,\Gamma_{\rho\lambda}^{\ \ \lambda})$$
$$+ (g^{\nu\alpha}\,\Gamma_{\mu\rho}^{\ \ \alpha} - g^{\alpha\rho}\,\Gamma_{\mu\rho}^{\ \ \nu})]\} \qquad (2\text{-}12)$$

As shown by Yilmaz [5] (p. 959, Appendix B), the symbolic manipulation programs MACSYMA and MATHEMATICA have been used to prove that Eq. 2-12 can be simplified to

$$U_\mu^{\ \nu} = (1/4\sqrt{[-g]})\,\partial_\alpha(\mathbf{g}^{\alpha\lambda}\,\mathbf{g}^{\nu\rho}(\partial_\rho\,\mathbf{g}_{\mu\lambda} - \partial_\lambda\,\mathbf{g}_{\mu\rho}) + \delta_\mu^{\ \nu}\,\partial_\beta\,\mathbf{g}^{\beta\alpha} - \delta_\mu^{\ \alpha}\,\partial_\beta\,\mathbf{g}^{\beta\nu} \qquad (2\text{-}13)$$

Bold symbols denote the following tensor densities:

$$\mathbf{g}^{\mu\nu} = \sqrt{[-g]}\,g^{\mu\nu} \quad , \quad \mathbf{g}_{\mu\nu} = g_{\mu\nu}/\sqrt{[-g]} \qquad (2\text{-}14)$$

Equation 2-4 gives a formula for 2 $\mathbf{U}_\mu^{\ \nu}$, and Eq. 2-9 shows that $u_\mu^{\ \nu}$ is equal to $-\tfrac{1}{2}\not{U}_\mu^{\ \nu}$. Hence $2\mathbf{U}_\mu^{\ \nu}$ is equal to

$$2\,\mathbf{U}_\mu^{\ \nu} = 2\sqrt{[g]}\,\not{U}_\mu^{\ \nu} = -4\sqrt{[-g]}\,u_\mu^{\ \nu} \qquad (2\text{-}15)$$

Combining Eqs 2-4, 2-15 gives

$$u_\mu^{\ \nu} = -(1/4\sqrt{[-g]})\{\Gamma_{ar}^{\ \ r}\,\partial_\mu\{g^{av}\sqrt{[-g]}\} - \Gamma_{rs}^{\ \ \nu}\,\partial_\mu\{g^{rs}\sqrt{[-g]}\} - \mathbf{G}\,\delta_\mu^{\ \nu} \qquad (2\text{-}16)$$

Substituting into this the expression for **G** of Eq. 2-5 gives:

$$u_\mu^{\ \nu} = (1/4\sqrt{[-g]}) \{ \Gamma_{rs}^{\ \ \nu}\, \partial_\mu\{g^{rs}\sqrt{[-g]}\} - \Gamma_{ar}^{\ \ r}\, \partial_\mu\{g^{a\nu}\sqrt{[-g]}\}$$

$$+ \tfrac{1}{4}\, \delta_\mu^{\ \nu}\, g^{ik}\, (\Gamma_{is}^{\ \ r}\, \Gamma_{kr}^{\ \ s} - \Gamma_{ik}^{\ \ r}\, \Gamma_{rs}^{\ \ s}) \qquad (2\text{-}17)$$

As shown by Yilmaz [5] (p. 959, Appendix B), the symbolic manipulation programs MACSYMA and MATHEMATICA have been used to prove that Eq. 2-17 can be simplified to

$$u_\mu^{\ \nu} = - W_\mu^{\ \nu} + \tfrac{1}{2}\, \delta_\mu^{\ \nu}\, W \qquad (2\text{-}18)$$

where $W_\mu^{\ \nu}$ is equal to

$$W_\mu^{\ \nu} = (1/8\sqrt{[-g]})\, \mathbf{g}^{\nu\lambda} \{ \partial_\lambda \mathbf{g}_{\alpha\beta}\, \partial_\mu \mathbf{g}^{\alpha\beta} - 2\, \partial_\lambda(\sqrt{[-g]})\, \partial_\mu(1/\sqrt{[-g]}) - 2\, \partial_\alpha \mathbf{g}_{\lambda\beta}\, \partial_\mu \mathbf{g}^{\alpha\beta} \} \qquad (2\text{-}19)$$

The scalar W in Eq. 2-18 is the trace of $W_\mu^{\ \nu}$, which is the sum of its diagonal elements:

$$W = W_0^{\ 0} + W_1^{\ 1} + W_2^{\ 2} + W_3^{\ 3} \qquad (2\text{-}20)$$

In Eq. 2-19, $\mathbf{g}^{\nu\lambda}/\sqrt{[-g]}$ can be replaced by $g^{\mu\lambda}$ to give

$$W_\mu^{\ \nu} = (1/8) g^{\nu\lambda}\, \partial_\lambda \mathbf{g}_{\alpha\beta}\, \partial_\mu \mathbf{g}^{\alpha\beta} - \tfrac{1}{4}\, g^{\nu\lambda}\, \partial_\lambda(\sqrt{[-g]})\, \partial_\mu(1/\sqrt{[-g]}) - \tfrac{1}{4}\, g^{\nu\lambda}\, \partial_\alpha \mathbf{g}_{\lambda\beta}\, \partial_\mu \mathbf{g}^{\alpha\beta} \qquad (2\text{-}21)$$

2,3 Freud Identity

Pauli [8] (p. 215, Eq. I) gives the following identity, which was first proven by P. Freud, and so we call it the *Freud identity*:

$$\partial_\nu[\mathbf{U}_\mu^{\ \nu} + \mathbf{G}_\mu^{\ \nu}] \equiv 0 \qquad (2\text{-}22)$$

Equation 2-10 shows that $[\mathbf{U}_\mu^{\ \nu} + \mathbf{G}_\mu^{\ \nu}]$ is equal to $-2\sqrt{[-g]}\, U_\mu^{\ \nu}$. Hence the Freud identity of Eq. 2-22 can be expressed as

$$\partial_\nu[\sqrt{[-g]}\, U_\mu^{\ \nu}] \equiv 0 \qquad (2\text{-}23)$$

As shown in *5,5 Addendum Chapter 5*, this identity has important implications in the Yilmaz theory.

2,4 Meaning of Yilmaz Pseudo Tensors

Equation 2-9 shows that the variable $u_\mu^{\ \nu}$ is equal to $-\tfrac{1}{2}\, \text{Ʉ}_\mu^{\ \nu}$ and Eq. 2-1 shows that $u_\mu^{\ \nu}$ is equal to $-8\pi \text{ŧ}_\mu^{\ \nu}$. Hence $u_\mu^{\ \nu}$ is equal to

$$u_\mu^{\ \nu} = -\tfrac{1}{2}\, \text{Ʉ}_\mu^{\ \nu} = 4\pi\, \text{ŧ}_\mu^{\ \nu} \qquad (2\text{-}24)$$

Thus, $u_\mu{}^\nu$ is proportional to the pseudo-tensor $ŧ_\mu{}^\nu$, which Einstein considered to represent the energy components of the gravitational field.

Combining Eqs. 2-8, 2-9 gives

$$U_\mu{}^\nu = -\tfrac{1}{2} G_\mu{}^\nu + u_\mu{}^\nu \tag{2-25}$$

This can be expressed as

$$-\tfrac{1}{2} G_\mu{}^\nu = U_\mu{}^\nu - u_\mu{}^\nu \tag{2-26}$$

The gravitational field equation for the Yilmaz theory is

$$-\tfrac{1}{2} G_\mu{}^\nu = \tau_\mu{}^\nu + t_\mu{}^\nu \tag{2-27}$$

Relating Eqs. 2-26, 2-27 shows that

$$\tau_\mu{}^\nu + t_\mu{}^\nu = U_\mu{}^\nu - u_\mu{}^\nu \tag{2-28}$$

By Eq. 2-24, the pseudo tensor $u_\mu{}^\nu$ is related to the energy in the gravitational field, and so is related to the stress-energy tensor $t_\mu{}^\nu$ for the gravitational field. It will be seen that the pseudo-tensor $U_\mu{}^\nu$ is related to the stress energy tensor $\tau_\mu{}^\nu$ of matter. These pseudo-tensors can be expressed as follows

$$U_\mu{}^\nu = \tau_\mu{}^\nu + z_\mu{}^\nu \tag{2-29}$$

$$u_\mu{}^\nu = -t_\mu{}^\nu + z_\mu{}^\nu \tag{2-30}$$

The variable $z_\mu{}^\nu$ is a non-tensor. In the combined expression ($U_\mu{}^\nu - u_\mu{}^\nu$), this non-tensor $z_\mu{}^\nu$ is canceled, and so the combined expression is a true tensor.

The concept of harmonic coordinates is discussed in Section 1. The harmonic coordinate condition greatly simplifies the analysis. It permits a wave to propagate in a single direction, without having a compensating wave in the reverse direction. Hence it allows energy to be radiated. These conditions can apply to electromagnetic waves and to gravitational waves.

It is shown in Section 1 that in harmonic coordinates the non-tensor $z_\mu{}^\nu$ in Eqs. 2-29, 2-30 is zero. Therefore, in harmonic coordinates the Yilmaz pseudo-tensors $U_\mu{}^\nu$ and $u_\mu{}^\nu$ are related as follows to the stress-energy tensors for matter and for the gravitational field:

$$U_\mu{}^\nu = \tau_\mu{}^\nu \qquad \textit{harmonic coordinates} \tag{2-31}$$

$$u_\mu{}^\nu = -t_\mu{}^\nu \qquad \textit{harmonic coordinates} \tag{2-32}$$

By Eq. 2-24, $u_\mu{}^\nu$ is equal to $4\pi\, ŧ_\mu{}^\nu$. Applying this to Eq. 2-30 gives

$$4\pi\, ŧ_{\mu}^{\nu} = - t_{\mu}^{\nu} + z_{\mu}^{\nu} \qquad (2\text{-}33)$$

This shows that the Einstein pseudo tensor $ŧ_{\mu}^{\nu}$ is not a true tensor because it is corrupted by the non-tensor component z_{μ}^{ν}. However in harmonic coordinates, the Einstein tensor $ŧ_{\mu}^{\nu}$ becomes a true tensor and is proportional to the stress energy tensor for the gravitational field t_{μ}^{ν}.

Applying Eq. 2-29 to Eq. 2-23 shows that the Freud identity can be expressed as

Freud identity: $\partial_{\nu}[\sqrt{[-g]}\, U_{\mu}^{\nu}] \equiv \partial_{\nu}[\sqrt{[-g]}(\tau_{\mu}^{\nu} + z_{\mu}^{\nu})] \equiv 0$ (2-34)

Since τ_{μ}^{ν}, z_{μ}^{ν} are independent, this identity must apply to each component separately. This gives the two separate identities:

$$\partial_{\nu}[\sqrt{[-g]}\, \tau_{\mu}^{\nu}] \equiv 0 \qquad (2\text{-}35)$$

$$\partial_{\nu}[\sqrt{[-g]}\, z_{\mu}^{\nu}] \equiv 0 \qquad (2\text{-}36)$$

Equation 2-35 is an important requirement, because it guarantees, in the Yilmaz theory, the conservation of energy and momentum for matter and electromagnetic fields.

3, Tensor Formulas Related to General Yilmaz Theory

This section calculates some important tensor formulas applicable to the general time-varying Yilmaz theory, which are used in Section 1.

3.1 Basic Differential Metric Equation for General Yilmaz Theory

As shown by Yilmaz [7] (p. 497, Appendix A), the general time-varying Yilmaz theory is defined by the following differential metric equation:

$$dg_{\mu\nu} = 2g_{\mu\nu}\, d\phi - 2\, g_{\mu\alpha}\, d\phi_{\nu}^{\alpha} - 2\, g_{\alpha\nu}\, d\phi_{\mu}^{\alpha} \qquad (3\text{-}1)$$

The scalar variable ϕ is the trace of the gravitational potential tensor ϕ_{μ}^{ν}, and so is equal to

$$\phi = \phi_0^{0} + \phi_1^{1} + \phi_2^{2} + \phi_3^{3} \qquad (3\text{-}2)$$

Section 3.2 will prove the following relation [xx]:

$$g_{\mu\alpha}\, d\phi_{\nu}^{\alpha} = g_{\alpha\nu}\, d\phi_{\mu}^{\alpha} \qquad (3\text{-}3)$$

Therefore, Eq. 3-1 can also be expressed in the following simplified forms:

$$dg_{\mu\nu} = 2g_{\mu\nu}\, d\phi - 4\, g_{\mu\alpha}\, d\phi_{\nu}^{\alpha} \qquad (3\text{-}4)$$

$$dg_{\mu\nu} = 2g_{\mu\nu}\, d\phi - 4\, g_{\alpha\nu}\, d\phi_\mu{}^\alpha \qquad (3\text{-}5)$$

Equation 3-4 was shown by Yilmaz [5] (p. 947).

3,2 Simplification of Differential Metric Equation

This section proves the relation given in Eq. 3-3. As was shown in *5,5 Addendum Chapter 5*, the definition for the gravitational potential tensor $\phi_\mu{}^\nu$ is

$$\phi_\mu{}^\nu = \Sigma\, [u_\mu\, u^\nu\, (\Delta M/r)\,]_{retarded} \qquad (3\text{-}6)$$

In the following discussion, the "retarded" notation is dropped but is still assumed. Let us take the derivative of this expression.

The Yilmaz theory is a microscopic theory of matter, rather than a macroscopic theory. The theory assumes that matter consists of a collection of point particles. This assumption is essential if the resultant theory is to be quantized and thereby made consistent with quantum mechanics. When the derivative $d\phi_\mu{}^\nu$ is calculated for a particle, the only variable in Eq. 3-6 that is differentiated is the distance R from the particle to the point of interest. The other quantities of Eq. 3-6 are constants relative to this differentiation.

For a single point particle, Eq. 3-6 becomes

$$\phi_\mu{}^\nu = u_\mu\, u^\nu\, (\Delta M/r) \qquad (3\text{-}7)$$

Differentiate this expression, treating r as the only variable:

$$d\phi_\mu{}^\nu = u_\mu\, u^\nu\, (\Delta M/r)(- dr/r) = \phi_\mu{}^\nu\, (- dr/r) \qquad (3\text{-}8)$$

For a sum of k point particles, Eq. 3-6 can be expressed as

$$\phi_\mu{}^\nu = \Sigma\, (\phi_\mu{}^\nu)_k \qquad (3\text{-}9)$$

where $(\phi_\mu{}^\nu)_k$ is the element of the tensor for a single particle. In accordance with Eq. 3-9, the derivative of this is

$$d\phi_\mu{}^\nu = \Sigma\, (\phi_\mu{}^\nu)_k\, (- dr_k/r_k) \qquad (3\text{-}10)$$

Hence the expression $g_{\mu\alpha}\, d\phi_\nu{}^\alpha$ is equal to

$$g_{\mu\alpha}\, d\phi_\nu{}^\alpha = \Sigma\, g_{\mu\alpha}\, (\phi_\nu{}^\alpha)_k\, (- dr_k/r_k) \qquad (3\text{-}11)$$

Apply the following relation to Eq. 3-11:

$$g_{\mu\alpha}\, \phi_\nu{}^\alpha = \phi_{\nu\mu} \qquad (3\text{-}12)$$

Equation 3-11 becomes

$$g_{\mu\alpha}\, d\phi_{\nu}{}^{\alpha} = \Sigma\, (\phi_{\nu\mu})_k\, (-\, dr_k/r_k) \qquad (3\text{-}13)$$

In like manner it can be shown that $g_{\nu\alpha}\, d\phi_{\mu}{}^{\alpha}$ is equal to

$$g_{\nu\alpha}\, d\phi_{\mu}{}^{\alpha} = \Sigma\, (\phi_{\mu\nu})_k\, (-\, dr_k/r_k) \qquad (3\text{-}14)$$

It is required that the gravitational potential tensor $\phi_{\mu}{}^{\alpha}$ be symmetric in its covariant and contravariant forms:

$$\phi_{\mu\nu} = \phi_{\nu\mu} \quad ; \quad \phi^{\mu\nu} = \phi^{\nu\mu} \qquad (3\text{-}15)$$

Consequently, the right-hand sides of Eqs. 3-14, 3-15 are equal. Equating the left-hand sides gives

$$g_{\mu\alpha}\, d\phi_{\nu}{}^{\alpha} = g_{\nu\alpha}\, d\phi_{\mu}{}^{\alpha} \qquad (3\text{-}16)$$

Since the metric tensor is symmetric, $g_{\nu\alpha}$ is equal to $g_{\nu\alpha}$. Hence Eq. 3-16 can be expressed as

$$g_{\mu\alpha}\, d\phi_{\nu}{}^{\alpha} = g_{\alpha\nu}\, d\phi_{\mu}{}^{\alpha} \qquad (3\text{-}16)$$

This is the same as the relation of Eq. 3-3, which was to be proven.

3,3 General Yilmaz Formula for the Metric Tensor Determinant g

The following analysis derives a simple formula for the determinant g of the metric tensor $g_{\mu\nu}$ for the general Yilmaz theory. Tolman [10] (p. 496, Eq. 39) gives the following general formula for the determinant g:

$$dg/g = g^{\mu\nu}\, dg_{\mu\nu} \qquad (3\text{-}17)$$

To apply this, multiply the expression for $dg_{\mu\nu}$ in Eq. 3-1 by $g^{\mu\nu}$ to obtain

$$g^{\mu\nu}\, dg_{\mu\nu} = 2g^{\mu\nu}\, g_{\mu\nu}\, d\phi - 2g^{\mu\nu}\, g_{\mu\alpha}\, d\phi_{\nu}{}^{\alpha} - 2g^{\mu\nu}\, g_{\alpha\nu}\, d\phi_{\mu}{}^{\alpha} \qquad (3\text{-}18)$$

The following general tensor formulas hold

$$g^{\mu\nu}\, g_{\mu\nu} = 4 \qquad (3\text{-}19)$$

$$g^{\mu\nu}\, g_{\mu\alpha} = \delta_{\alpha}{}^{\nu} \qquad (3\text{-}20)$$

$$g^{\mu\nu}\, g_{\alpha\nu} = \delta_{\alpha}{}^{\mu} \qquad (3\text{-}21)$$

Apply to Eq. 3-18 the formulas of Eqs. 3-19 to 3-21:

$$g^{\mu\nu}\, dg_{\mu\nu} \;=\; 8\, d\phi - 2\, \delta_\alpha{}^\nu\, d\phi_\nu{}^\alpha - 2\, \delta_\alpha{}^\mu\, d\phi_\mu{}^\alpha \qquad (3\text{-}22)$$

Since ϕ is the trace of $\phi_\mu{}^\nu$, the following tensor formulas hold

$$\delta_\alpha{}^\nu\, \phi_\nu{}^\alpha \;=\; \phi \qquad (3\text{-}23)$$

$$\delta_\alpha{}^\mu\, \phi_\mu{}^\alpha \;=\; \phi \qquad (3\text{-}24)$$

Apply to Eq. 3-22 the formulas of Eqs. 3-23, 3-24:

$$g^{\mu\nu}\, dg_{\mu\nu} \;=\; 8\, d\phi - 2\, d\phi - 2\, d\phi \;=\; 4\, d\phi \qquad (3\text{-}25)$$

Applying Eq. 3-25 into Eq. 3-17 gives

$$dg/g \;=\; 4\, d\phi \qquad (3\text{-}26)$$

The solution to this equation is

$$g \;=\; \pm\, e^{4\phi} \qquad (3\text{-}27)$$

We can prove that Eq. 3-27 is the solution to Eq. 3-24 by differentiating the expression for g in Eq. 3-27. The analysis does not give us the sign for g. However, we know that g must be negative. Hence the actual formulas for the determinant g is

$$g \;=\; -\, e^{4\phi} \qquad (3\text{-}28)$$

The expression ($\sqrt{[-g]}$), which is used in the tensor densities is equal to

$$\sqrt{[-g]} \;=\; e^{2\phi} \qquad (3\text{-}29)$$

Although Eqs. 3-28, 3-29 are very simple, they ***always hold*** for the Yilmaz theory, even for the general time-varying Yilmaz theory.

3,4 Formula for Derivative of Gravitational Potential Tensor $\boldsymbol{\phi_\mu{}^\nu}$

Yilmaz [5] (p. 959, App. B) gives the following formula, which will now be proven:

$$-\,4\, \partial_\lambda\, \phi_\mu{}^\nu \;=\; \mathbf{g}^{\nu\rho}\, \partial_\lambda\, \mathbf{g}_{\mu\rho} \qquad (3\text{-}30)$$

The metric tensor densities are defined as follows:

$$\mathbf{g}^{\mu\nu} \;=\; \sqrt{[-g]}\, g^{\mu\nu} \quad , \quad \mathbf{g}_{\mu\nu} \;=\; g_{\mu\nu}/\sqrt{[-g]} \qquad (3\text{-}31)$$

Apply to the right-hand expression of Eq. 3-30 the definitions for the metric tensor densities in Eqs. 3-31, and then apply the expression for $\sqrt{[-g]}$ in Eq. 3-29:

$$\mathbf{g}^{\nu\rho}\,\partial_\lambda\,\mathbf{g}_{\mu\rho} \;=\; \surd[-g]\; g^{\nu\rho}\,\partial_\lambda\{g_{\mu\rho}/\surd[-g]\} \;=\; e^{2\phi}\,g^{\nu\rho}\,\partial_\lambda\{e^{-2\phi}\,g_{\mu\rho}\} \qquad (3\text{-}32)$$

Expand the derivative

$$\mathbf{g}^{\nu\rho}\,\partial_\lambda\,\mathbf{g}_{\mu\rho} \;=\; e^{2\phi}\,g^{\nu\rho}\,\{\,e^{-2\phi}\,\partial_\lambda\,g_{\mu\rho} + g_{\mu\rho}\,\partial_\lambda(e^{-2\phi})\,\} \;=\; e^{2\phi}\,g^{\nu\rho}\,\{e^{-2\phi}\,\partial_\lambda\,g_{\mu\rho} - 2\,g_{\mu\rho}\,e^{-2\phi}\,\partial_\lambda\phi\,\}$$

$$=\; g^{\nu\rho}\,\partial_\lambda g_{\mu\rho} - 2\,g^{\nu\rho}\,g_{\mu\rho}\,\partial_\lambda\phi \qquad (3\text{-}33)$$

Consider the following identities

$$g^{\nu\rho}\,g_{\mu\rho} \;=\; \delta_\mu{}^\nu \qquad ; \qquad g^{\nu\rho}\,g_{\alpha\rho} \;=\; \delta_\alpha{}^\nu \qquad (3\text{-}34)$$

Applying the first identity to Eq. 3-33 gives

$$\mathbf{g}^{\nu\rho}\,\partial_\lambda\,\mathbf{g}_{\mu\rho} \;=\; g^{\nu\rho}\,\partial_\lambda\,g_{\mu\rho} - 2\,\delta_\mu{}^\nu\,\partial_\lambda\phi \qquad (3\text{-}35)$$

In Eq. 3-35, substitute for ($\partial_\lambda\, g_{\mu\rho}$) the simplified differential metric formula for the Yilmaz theory that is given in Eq. 3-5:

$$\mathbf{g}^{\nu\rho}\,\partial_\lambda\,\mathbf{g}_{\mu\rho} \;=\; g^{\nu\rho}\,\{\,2g_{\mu\rho}\,\partial_\lambda\phi - 4\,g_{\alpha\rho}\,\partial_\lambda\phi_\mu{}^\alpha\,\} - 2\,\delta_\mu{}^\nu\,\partial_\lambda\phi$$

$$=\; 2\,g^{\nu\rho}\,g_{\mu\rho}\,\partial_\lambda\,\phi - 4\,g^{\nu\rho}\,g_{\alpha\rho}\,\partial_\lambda\,\phi_\mu{}^\alpha - 2\,\delta_\mu{}^\nu\,\partial_\lambda\,\phi \qquad (3\text{-}36)$$

Apply to the first and second terms the first and second identities of Eq. 3-34. This gives

$$\mathbf{g}^{\nu\rho}\,\partial_\lambda\,\mathbf{g}_{\mu\rho} \;=\; 2\,\delta_\mu{}^\nu\,\partial_\lambda\,\phi - 4\,\delta_\alpha{}^\nu\,\partial_\lambda\,\phi_\mu{}^\alpha - 2\,\delta_\mu{}^\nu\,\partial_\lambda\,\phi \;=\; -\,4\,\delta_\alpha{}^\nu\,\partial_\lambda\,\phi_\mu{}^\alpha \qquad (3\text{-}37)$$

This is zero except when α is equal to ν. Setting α equal to ν gives

$$\mathbf{g}^{\nu\rho}\,\partial_\lambda\,\mathbf{g}_{\mu\rho} \;=\; -\,4\,\partial_\lambda\,\phi_\mu{}^\nu \qquad (3\text{-}38)$$

This agrees with Eq. 3-30, which was to be proven. A related formula can be developed as follows by expanding the differentiation of Eq. 3-38:

$$-\,4\partial_\lambda\,\phi_\mu{}^\nu \;=\; \mathbf{g}^{\nu\rho}\,\partial_\lambda\,\mathbf{g}_{\mu\rho} \;=\; \partial_\lambda(\mathbf{g}^{\nu\rho}\,\mathbf{g}_{\mu\rho}) - \mathbf{g}_{\mu\rho}\,\partial_\lambda\,\mathbf{g}^{\nu\rho} \;=\; \partial_\lambda(\delta_\mu{}^\nu) - \mathbf{g}_{\mu\rho}\,\partial_\lambda\,\mathbf{g}^{\nu\rho} \qquad (3\text{-}39)$$

The first identity of Eq. 3-34 was applied to the first term. Since the derivative $\partial_\lambda\delta_\mu{}^\nu$ is zero, Eq. 3-39 reduces to

$$\mathbf{g}_{\mu\rho}\,\partial_\lambda\,\mathbf{g}^{\nu\rho} \;=\; 4\,\partial_\lambda\,\phi_\mu{}^\nu \qquad (3\text{-}40)$$

Multiplying Eqs 3-38, 3-40 together gives

$$\mathbf{g}^{\beta\rho}\,\partial_\lambda\,\mathbf{g}_{\alpha\rho}\,(\mathbf{g}_{\beta\rho}\,\partial_\mu\,\mathbf{g}^{\alpha\rho}\,) \;=\; -\,16\,\partial_\lambda\,\phi_\alpha{}^\beta\,\partial_\mu\,\phi_\beta{}^\alpha \;=\; \partial_\lambda\,\mathbf{g}_{\alpha\rho}\,\partial_\mu\,\mathbf{g}^{\alpha\rho} \qquad (3\text{-}41)$$

The product ($\mathbf{g}^{\beta\rho}\,\mathbf{g}_{\beta\rho}$) is unity, and so Eq. 3-41 simplifies to

$$\partial_\lambda \mathbf{g}_{\alpha\rho}\, \partial_\mu \mathbf{g}^{\alpha\rho} = -16\, \partial_\lambda\, \phi_\alpha{}^\beta\, \partial_\mu\, \phi_\beta{}^\alpha \qquad (3\text{-}42)$$

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